Unifying Points, Beams, and Paths in Volumetric Light Transport Simulation
Supplemental Document

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1 Variance Analysis

Fig.1 shows the normalized standard deviation (NSD) for all the 25 estimators. This is a superset of the graphs shown in Fig. 6 of the paper.

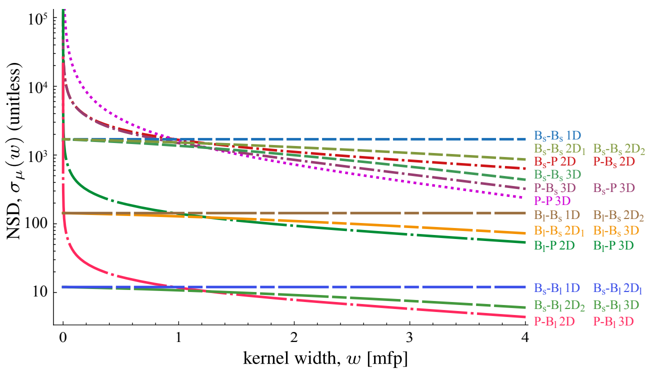

Figure 1: *Normalized standard deviation (NSD) as a function of the kernel width for all 25 estimators.*

2 Extended Balance Heuristic: Derivation

Proof of Theorem 1 from the paper. We follow the optimality proof of the balance heuristic [Veach 1997, Appendix 9.A] while adjusting it for the estimator F^{C} given by Equation (40) in the paper. We first give a sketch of the proof. The variance of F^{C} is written as $V[F^{\mathrm{C}}] = \mathrm{A} - \mathrm{B}$. Both A and B depend on the choice of weighting functions. Finding weighting functions that minimize A − B is difficult, so we follow Veach [1997] and proceed as follows:

1. Find weighting functions that minimize A. The result is the extended balance heuristic, Equation (42) in the paper, so no other set of weighting functions can yield smaller A.

2. Derive lower and upper bounds on B that hold for any set of weighting functions w_i. This provides the variance bound in Theorem 1.

The variance of the combined estimator F^{C} can be written as

$$V[F^{\mathrm{C}}] = V\left[\sum_{i=1}^{n}\frac{1}{n_i}\sum_{j=1}^{n_i}F^{\mathrm{C}}_{i,j}\right] = \sum_{i=1}^{n}\frac{1}{n_i^2}\sum_{j=1}^{n_i}V\left[F^{\mathrm{C}}_{i,j}\right]$$
$$= \underbrace{\left(\sum_{i=1}^{n}\frac{1}{n_i^2}\sum_{j=1}^{n_i}E\left[\left(F^{\mathrm{C}}_{i,j}\right)^2\right]\right)}_{\mathrm{A}} - \underbrace{\left(\sum_{i=1}^{n}\frac{1}{n_i^2}\sum_{j=1}^{n_i}E\left[F^{\mathrm{C}}_{i,j}\right]^2\right)}_{\mathrm{B}}$$

Term A.

$$\mathrm{A} = \sum_{i=1}^{u}\frac{1}{n_i}E\left[\left(w_i(X_{i,j})\frac{f(X_{i,j})}{p_i(X_{i,j})}\right)^2\right] + \sum_{i=u+1}^{n}\frac{1}{n_i}E\left[\left(w_i(X_{i,j})\frac{f_i(X_{i,j},Y_{i,j})}{p_i(X_{i,j},X_{i,j})}\right)^2\right]$$
$$= \sum_{i=1}^{u}\frac{1}{n_i}\int_{\mathcal{D}_x}\left(w_i(x)\frac{f(x)}{p_i(x)}\right)^2 p_i(x)\,\mathrm{d}x + \sum_{i=u+1}^{n}\frac{1}{n_i}\int_{\mathcal{D}_x}\int_{\mathcal{D}_{y_i}}\left(w_i(x)\frac{f_i(x,y_i)}{p_i(x,y_i)}\right)^2 p_i(x,y_i)\,\mathrm{d}y_i\,\mathrm{d}x$$
$$= \int_{\mathcal{D}_x}\left(\sum_{i=1}^{u}\frac{w_i^2(x)}{n_i}\frac{f^2(x)}{p_i(x)} + \sum_{i=u+1}^{n}\frac{w_i^2(x)}{n_i}\left[\int_{\mathcal{D}_{y_i}}\frac{f_i^2(x,y_i)}{p_i(x,y_i)}\mathrm{d}y_i\right]\right)\mathrm{d}x$$
$$= \int_{\mathcal{D}_x}\sum_{i=1}^{n}\frac{w_i^2(x)\kappa_i(x)}{n_i}\,\mathrm{d}x,$$

where $\kappa_i(x)$ is given by Equation (43) in the paper. We want to find the weighting functions that minimize A, subject to $\sum_{i=1}^{n} w_i(x) = 1$ for any x. Performing a point-wise minimization as in [Veach 1997, Appendix 9.A] yields the extended balance heuristic, Equation (42) in the paper. No other set of weighting functions can make the term A smaller.

Term B. To derive the desired bounds on the term B, we first let

$$f_i(x) = \begin{cases} f(x) & \text{if } 1 \le i \le u \\ \int_{\mathcal{D}_{y_i}} f_i(x,y_i)\,\mathrm{d}y_i & \text{if } u < i \le n. \end{cases}$$
$$\mu_i \equiv E[F^{\mathrm{C}}_{i,j}] = \int_{\mathcal{D}_x} w_i(x) f_i(x)\,\mathrm{d}x,$$

and also

$$\begin{aligned} f^+(x) &= \max_i f_i(x) & f^-(x) &= \min_i f_i(x) \\ b^+(x) &= f^+(x) - f(x) & b^-(x) &= f(x) - f^-(x) \\ \mu_i^+ &= \int_{\mathcal{D}_x} w_i(x) f^+(x)\,\mathrm{d}x & \mu_i^- &= \int_{\mathcal{D}_x} w_i(x) f^-(x)\,\mathrm{d}x \\ \beta^+ &= \int_{\mathcal{D}_x} b^+(x)\,\mathrm{d}x & \beta^- &= \int_{\mathcal{D}_x} b^-(x)\,\mathrm{d}x \\ \mu^+ &= \int_{\mathcal{D}_x} f^+(x)\,\mathrm{d}x = I + \beta^+ & \mu^- &= \int_{\mathcal{D}_x} f^-(x)\,\mathrm{d}x = I - \beta^-. \end{aligned} \tag{1}$$

Above, $f^-(x)$ and $f^+(x)$ are lower and upper bounds on the contribution of all techniques for any x, and β^- and β^+ can be interpreted as bounds on the bias of the combined estimator given by Equation (40) in the paper with any valid weighting heuristic. The upper bound of term B is derived as follows:

$$\mathrm{B} = \sum_{i=1}^{n} \frac{\mu_i^2}{n_i} \le \sum_{i=1}^{n} \frac{(\mu_i^+)^2}{n_i}$$
$$\le \frac{1}{\min_i n_i} \left(\sum_{i=1}^{n} \mu_i^+ \right)^2 = \frac{1}{\min_i n_i} \sum_{i=1}^{n} (\mu^+)^2.$$

To derive the lower bound of term B, we write

$$\mathrm{B} = \sum_{i=1}^{n} \frac{\mu_i^2}{n_i} \ge \sum_{i=1}^{n} \frac{(\mu_i^-)^2}{n_i}. \tag{2}$$

We want to minimize the above expression subject to $\sum_i \mu_i^- = \mu^-$. The method of Lagrange multipliers [Veach 1997, Appendix 9.A] yields the lower bound of

$$\frac{1}{\sum_{i=1}^{n} n_i} (\mu^-)^2. \tag{3}$$

References

VEACH, E. 1997. *Robust Monte Carlo methods for light transport simulation*. PhD thesis, Stanford, CA, USA.