Linear Transformations

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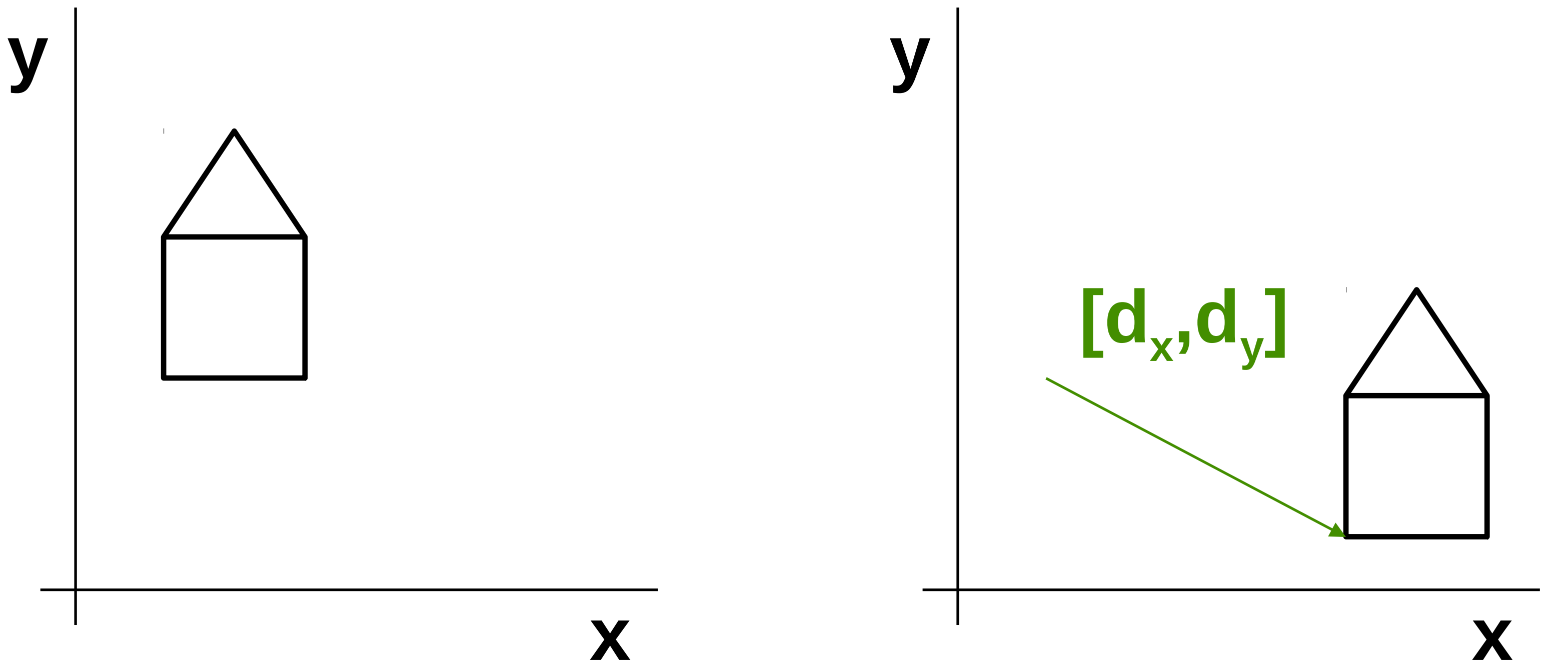
Requirements

- **Frequently used transformations:**
  - Move, rotate, zoom in/out, ...
  - Parallel and perspective projection
- Easy and **efficient implementation**
  - Lots of these transformations are computed (often up to 10^6 transformations in one go)
- **Special operations**
  - Concatenation of simple transforms, calculation of the inverse transform, ...

Displacement in the plane



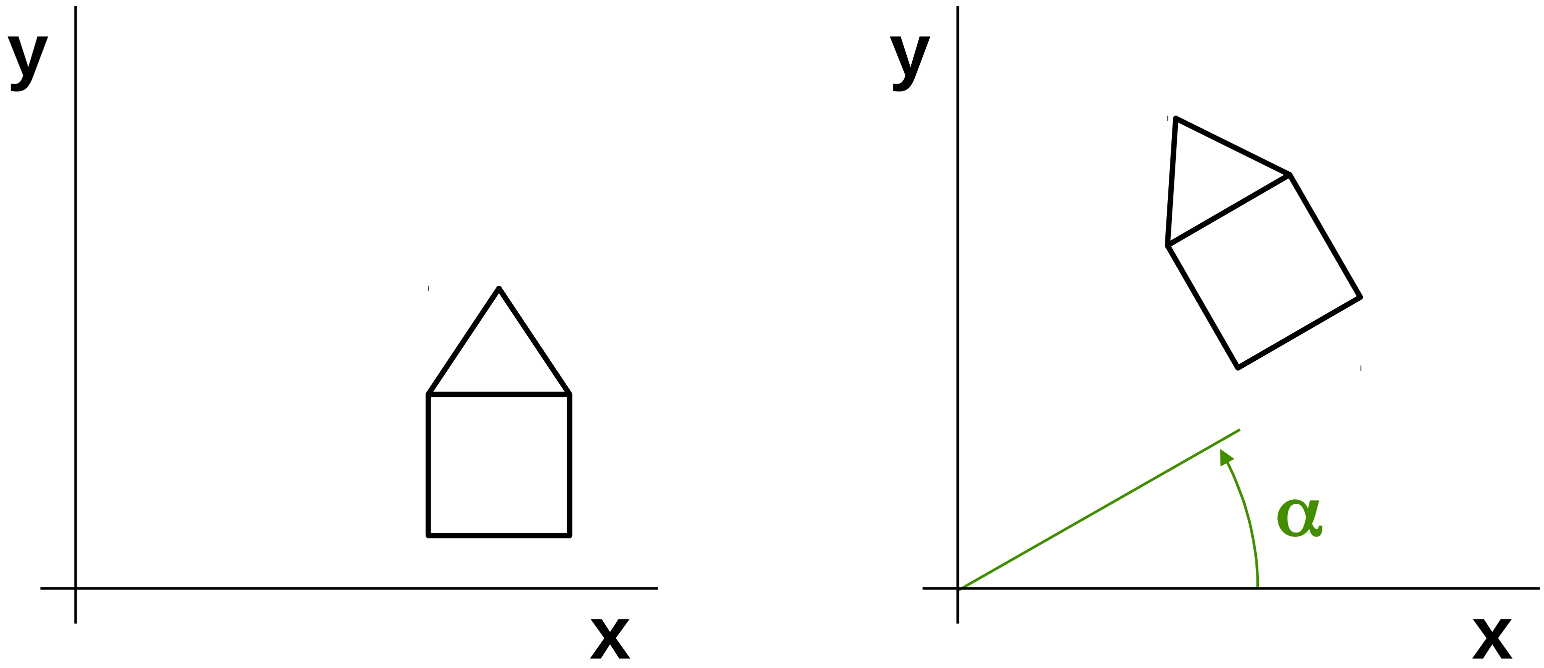
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} d_x & d_y \end{bmatrix}$$

Matrix Transformations

- Multiplications between points and matrices
  - Cartesian coordinates of the point **[x,y]** are a **row vector**
  - **Transformation matrices** are square (in the plane, 2×2)

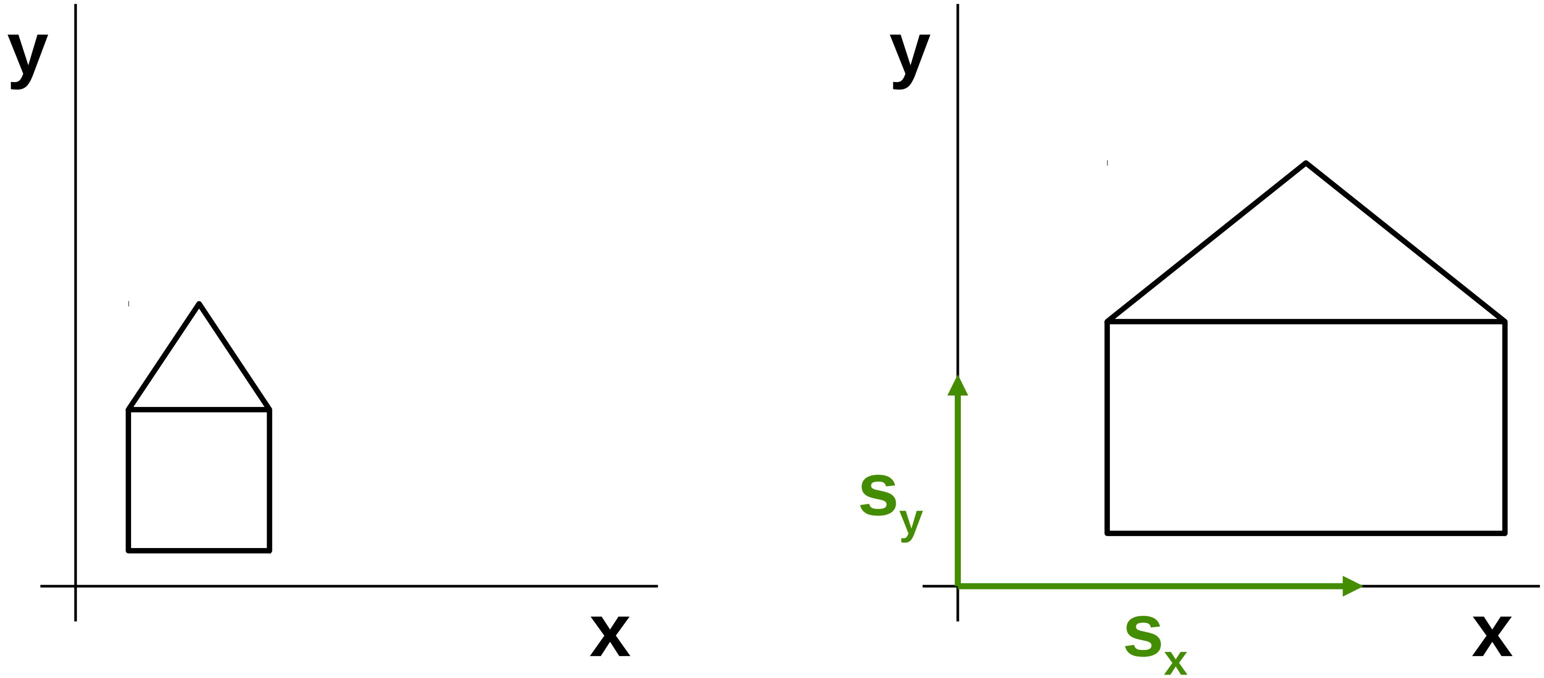
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

2D Rotation around the Origin



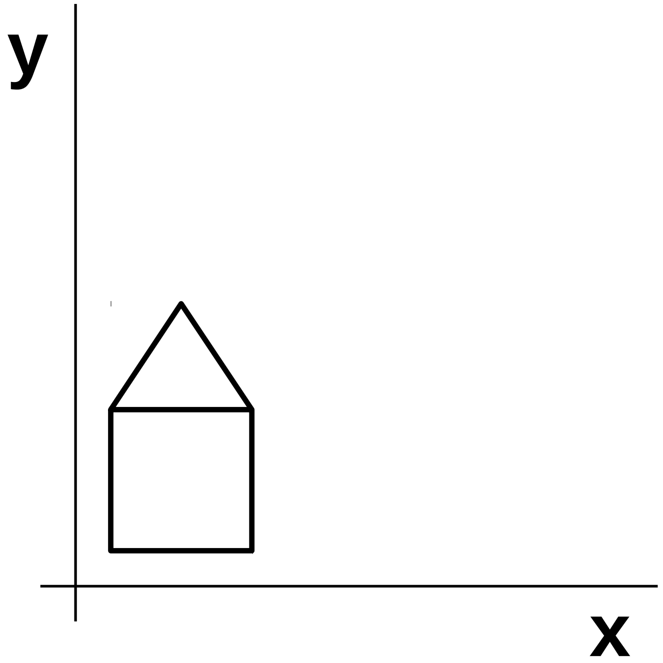
$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

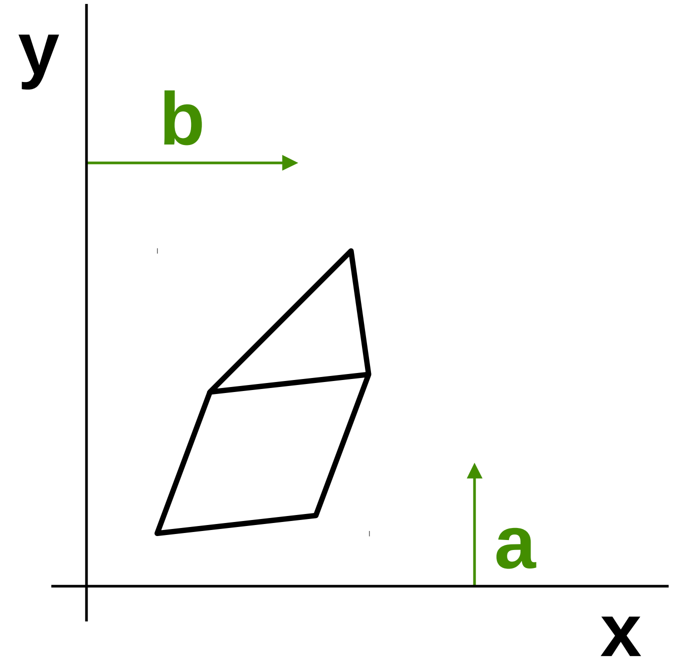
2D Shrink / Enlarge



$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

2D Shear





$$\mathbf{Sh(a,b)} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

Homogeneous Coordinates

- Unified representation of **affine transformations**
  - Transformations that retain straight lines
  - **Translation** in nD cannot be represented by a n×n matrix
- The most commonly used **non-affine transform:**
  - **Perspective transform** (projection)
- Representation of complex transforms
  - Matrix multiplication (associativity)

Algebraic Motivation

A line in the plane has coordinates **[a,b,c]** (not unambiguous):

$$a \cdot x + b \cdot y + c = 0,$$

A point in the plane has coordinates **[x,y]** (unambiguous).

Task 1: find the **line [a,b,c]** that goes through two fixed points **[x_1,y_1]** a **[x_2,y_2]**:

$$a \cdot x_1 + b \cdot y_1 + c = 0$$
$$a \cdot x_2 + b \cdot y_2 + c = 0$$

System **(1)**

Algebraic Motivation

Task 2: find the **point [x,y]**, where two lines $[a_1,b_1,c_1]$ and $[a_2,b_2,c_2]$ intersect:

$$a_1 \cdot x + b_1 \cdot y + c_1 = 0$$
$$a_2 \cdot x + b_2 \cdot y + c_2 = 0$$

System **(2)**

System (1) has infinitely many solutions, while **System (2)** only has a solution if $a_1 \cdot b_2 \neq a_2 \cdot b_1$

Algebraic Motivation

If one extends the plane by **points at infinity** and introduces **homogeneous coordinates [x,y,w]** the two preceding systems will be symmetrical, and system **(2')** will always be solveable:

$$\mathbf{a}\cdot x_1 + \mathbf{b}\cdot y_1 + \mathbf{c}\cdot w_1 = 0$$
$$\mathbf{a}\cdot x_2 + \mathbf{b}\cdot y_2 + \mathbf{c}\cdot w_2 = 0$$

System **(1')**

$$a_1\cdot \mathbf{x} + b_1\cdot \mathbf{y} + c_1\cdot \mathbf{w} = 0$$
$$a_2\cdot \mathbf{x} + b_2\cdot \mathbf{y} + c_2\cdot \mathbf{w} = 0$$

System **(2')**

Coordinate Conversions

Cartesian to homogeneous

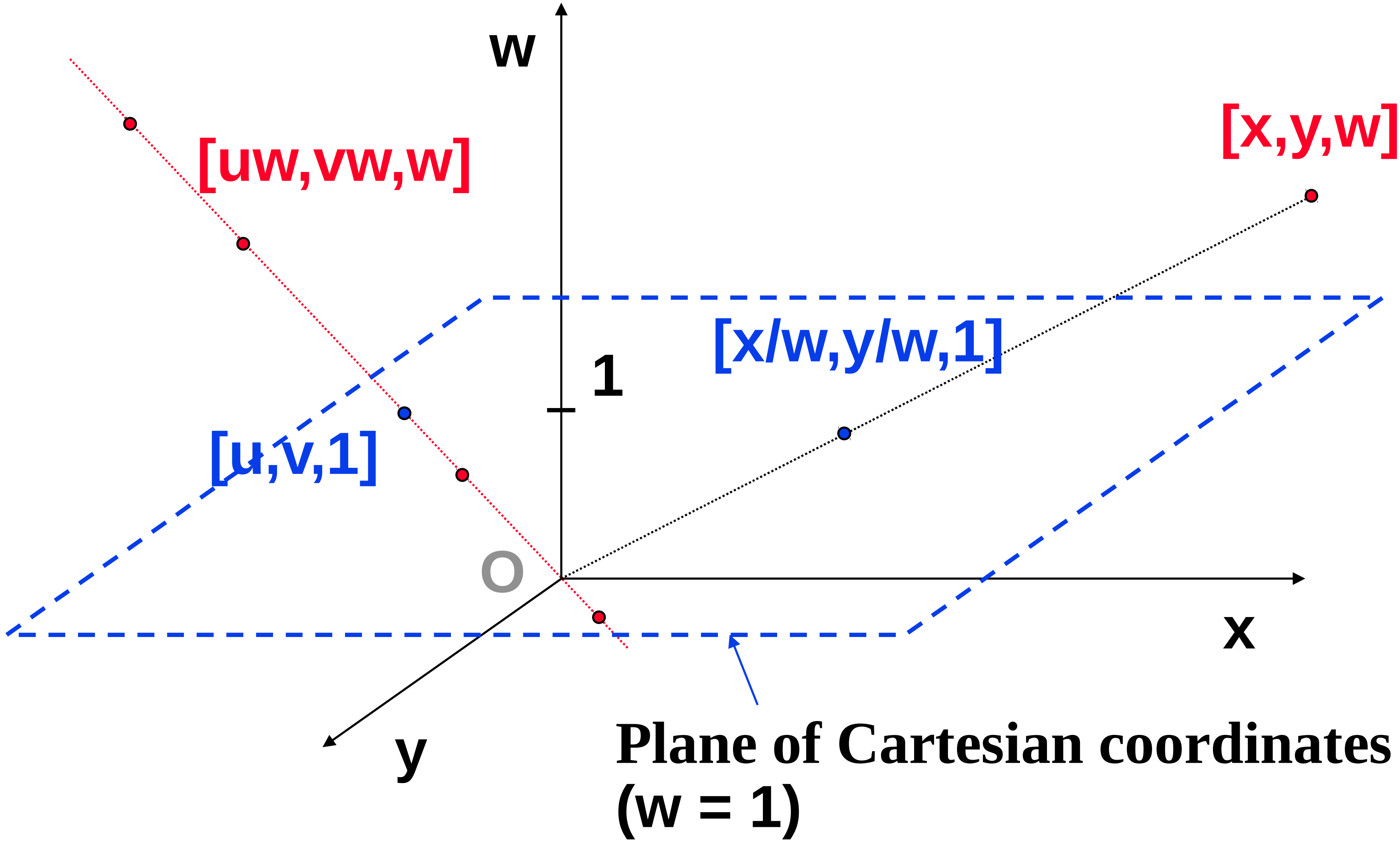
$$\begin{bmatrix} x & y \end{bmatrix} \rightarrow \begin{bmatrix} x & y & 1 \end{bmatrix}$$

Homogeneous to Cartesian (only finite points):

$$\begin{bmatrix} x & y & w \end{bmatrix} \rightarrow \begin{bmatrix} \frac{x}{w} & \frac{y}{w} \end{bmatrix}$$

$$w \neq 0$$

Geometric Interpretation



Homogeneous Transformation Matrices

Translation

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

Rotation

$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

Shear

$$\mathbf{Sh(a,b)} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite transformation:

$$\left(\left(\left([x,y,w]\cdot T_1\right)\cdot T_2\right)\cdot T_3\right) = [x,y,w]\cdot\left(T_1\cdot T_2\cdot T_3\right)$$

Rotation by an angle of α around point **[x,y]**:

$$R(x,y,\alpha) = T(-x,-y)\cdot R(\alpha)\cdot T(x,y)$$

Screen Transformations

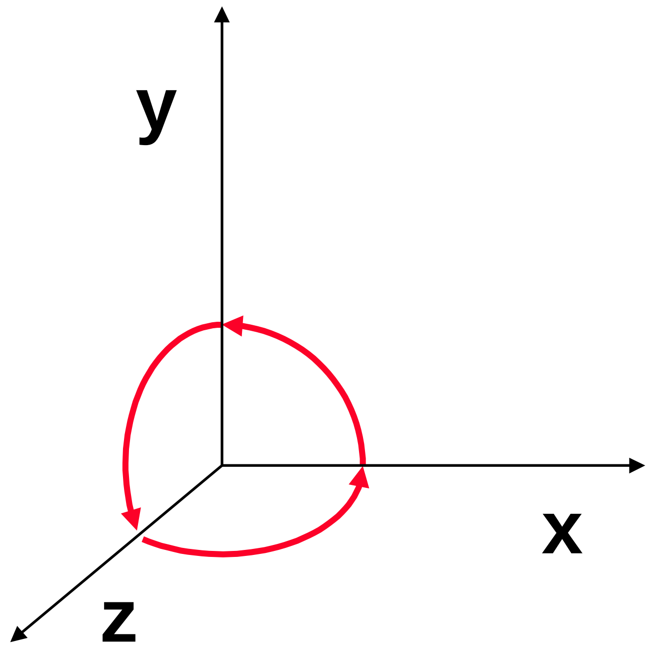
screen coordinate systems

Moving from general real coordinates to the **coordinate system of the display window:**

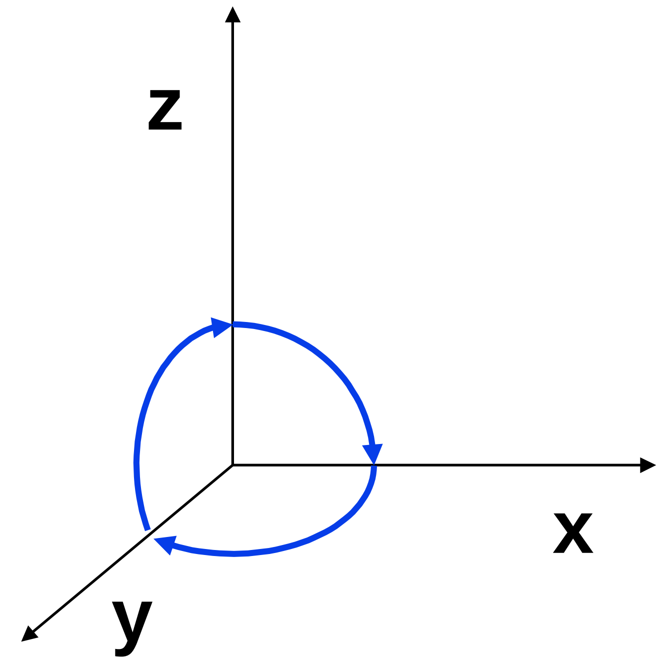
$$X_{int} = \mathrm{round}\,(\, D_x + S_x * X_f \,)$$

$$Y_{int} = \mathrm{round}\,(\, D_y + S_y * Y_f \,)$$

3D Coordinates



left-winding system
(„right-handed”)



right-winding system
(„left-handed”)

Homogeneous Coordinates

$$[x \quad y \quad z] \rightarrow [x \quad y \quad z \quad 1]$$

$$[x \quad y \quad z \quad w] \rightarrow \left[\frac{x}{w} \quad \frac{y}{w} \quad \frac{z}{w}\right] \qquad (w \neq 0)$$

Matrix transformations:

$$[x' \quad y' \quad z' \quad w'] = [x \quad y \quad z \quad w] \cdot \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}$$

Homogeneous Transformation Matrices

Translation

$$T(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

Shear

$$Sh(a, b, c, d, e, f) = \begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

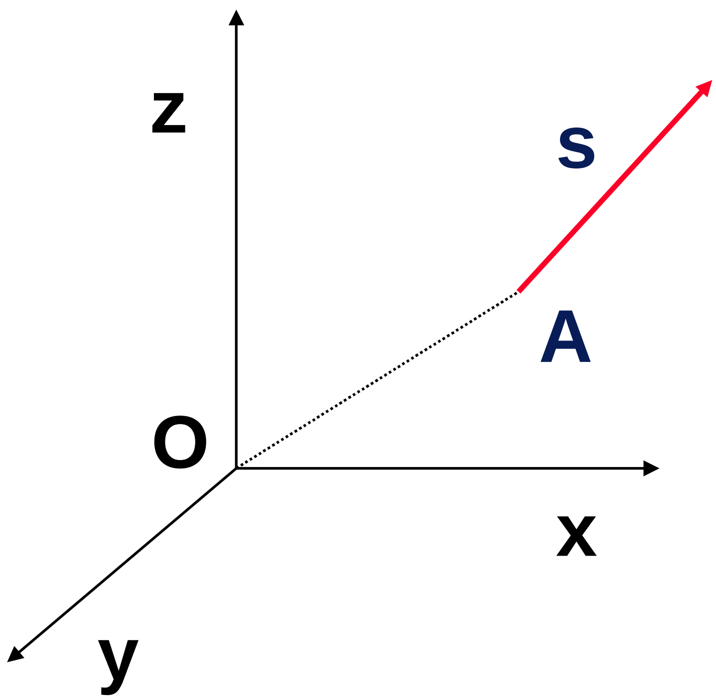
Rotation around the **y** axis

$$R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around the **z** axis

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Aligning a Ray with the **z** Axis



A ray is given by a point **A** and a direction vector **s**

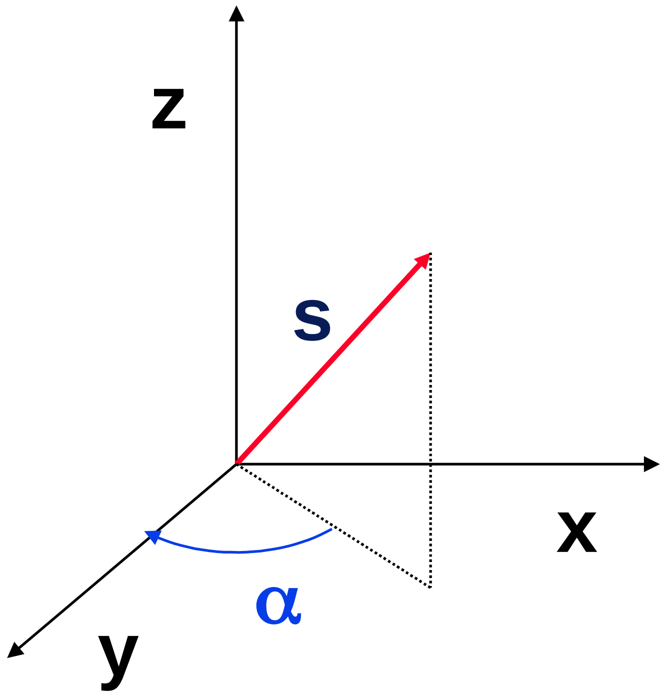
$$\mathbf{M} = \mathbf{T}(-\mathbf{A})$$

$$\mathbf{M}^{-1} = \mathbf{T}(\mathbf{A})$$

1. step:

transfer point **A** to the origin

Aligning a Ray with the **z** Axis



$$M = T(-A) \cdot R_z(\alpha)$$

$$M^{-1} = R_z(-\alpha) \cdot T(A)$$

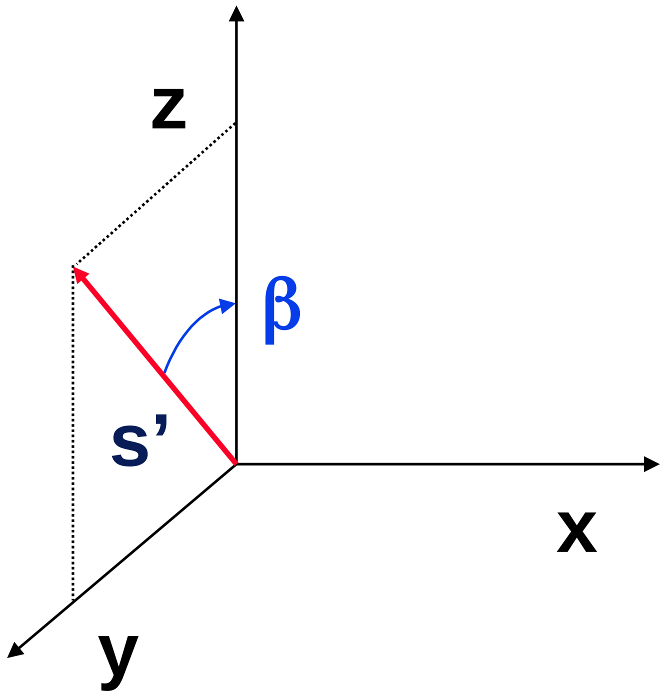
$$\cos\alpha = \frac{s_y}{\sqrt{s_x^2 + s_y^2}}$$

$$\sin\alpha = \frac{s_x}{\sqrt{s_x^2 + s_y^2}}$$

2. step:

rotation of the vector to the **yz** plane (around **z**)

Aligning a Ray with the **z** Axis



$$M = T(-A) \cdot R_z(\alpha) \cdot R_x(\beta)$$

$$M^{-1} = R_x(-\beta) \cdot R_z(-\alpha) \cdot T(A)$$

$$\cos\beta = \frac{s_z}{\sqrt{s_x^2 + s_y^2 + s_z^2}}$$

$$|\sin\beta| = \frac{\sqrt{s_x^2 + s_y^2}}{\sqrt{s_x^2 + s_y^2 + s_z^2}}$$

3. step:

rotation of the vector to the **z** axis (around **x**)

Applying a Transformation **M**

$$M(A,s) = T(-A) \cdot R_z(\alpha) \cdot R_x(\beta)$$

$$M(A,s)^{-1} = R_x(-\beta) \cdot R_z(-\alpha) \cdot T(A)$$

Rotation around an axis:

$$R(A,s,\theta) = M(A,s) \cdot R_z(\theta) \cdot M(A,s)^{-1}$$

Mirroring with respect to a plane:

$$Mirror(A,n) = M(A,n) \cdot S(1,1,-1) \cdot M(A,n)^{-1}$$

Computing the Inverse Transform

1. Matrix inversion: $\mathbf{M}^{-1}$

2. Stepwise:

$$\mathbf{M} = \mathbf{A} \cdot \mathbf{B} \cdot \boxed{\mathbf{C}}$$

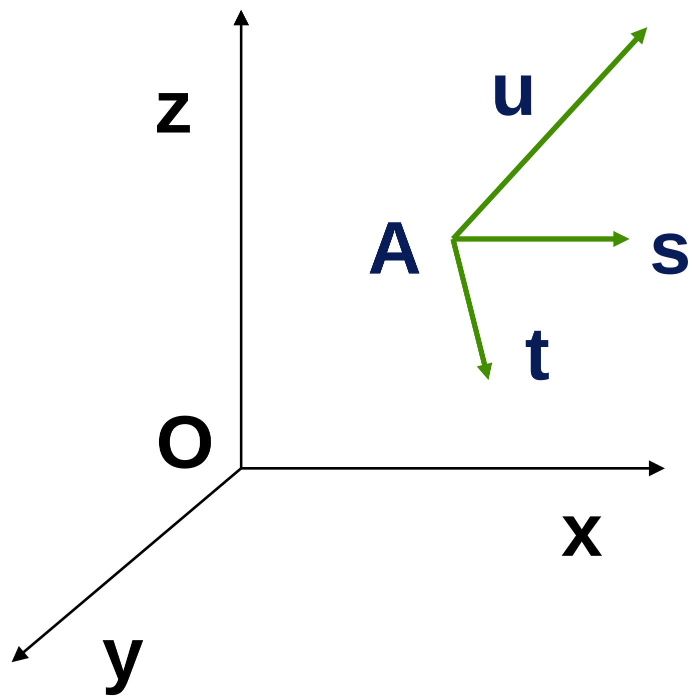
$$\mathbf{M}^{-1} = \boxed{\mathbf{C}^{-1}} \cdot \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

3. Transposition (orthonormal matrix):

$\mathbf{R}^{-1} = \mathbf{R}^{T}$ **for orthonormal matrices R**

(orthonormal matrices are e.g. all **rotation matrices**)

Conversion between Coordinate Systems



A **coordinate system** is given by its origin **A** and three basis vectors **s, t, u**

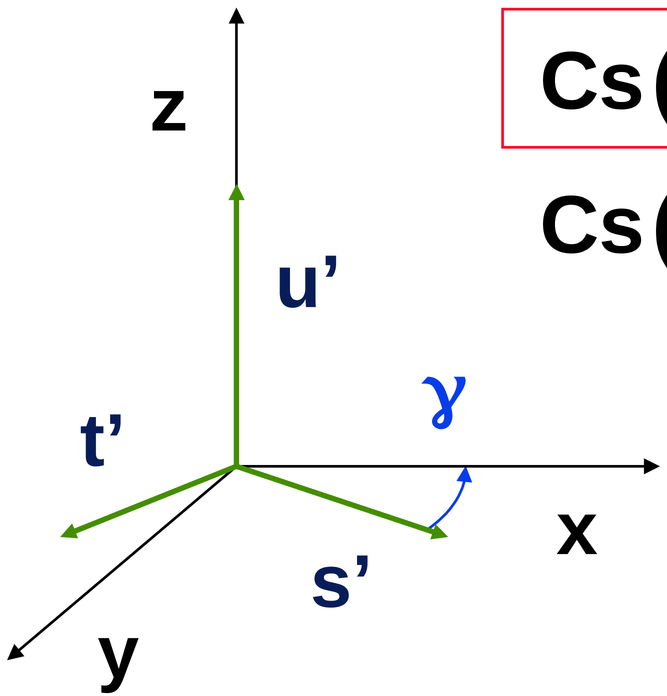
$$Cs = M(A, u)$$

$$Cs^{-1} = M(A, u)^{-1}$$

1. step:

Align ray **(A,u)** with the **z** axis

Conversion between Coordinate Systems



$$Cs(A,s,t,u) = M(A,u) \cdot R_z(\gamma)$$

$$Cs(A,s,t,u)^{-1} = R_z(-\gamma) \cdot M(A,u)^{-1}$$

$$\cos\gamma = \frac{|s \cdot M(A,u)|_x}{|s \cdot M(A,u)|}$$

$$\sin\gamma = \frac{|s \cdot M(A,u)|_y}{|s \cdot M(A,u)|}$$

2. step:

rotation of the axis **s'→x** and **t'→y** (around **z=u'**)

End

Further information:

- **J. Foley, A. van Dam, S. Feiner, J. Hughes**: ***Computer Graphics, Principles and Practice***, 201-227
- **Jiří Žára a kol.**: ***Počítačová grafika***, principy a algoritmy, 73-84