Ray × scene intersections

CGG MFF UK Praha

pepca@cgg.mff.cuni.cz

http://cgg.mff.cuni.cz/~pepca/

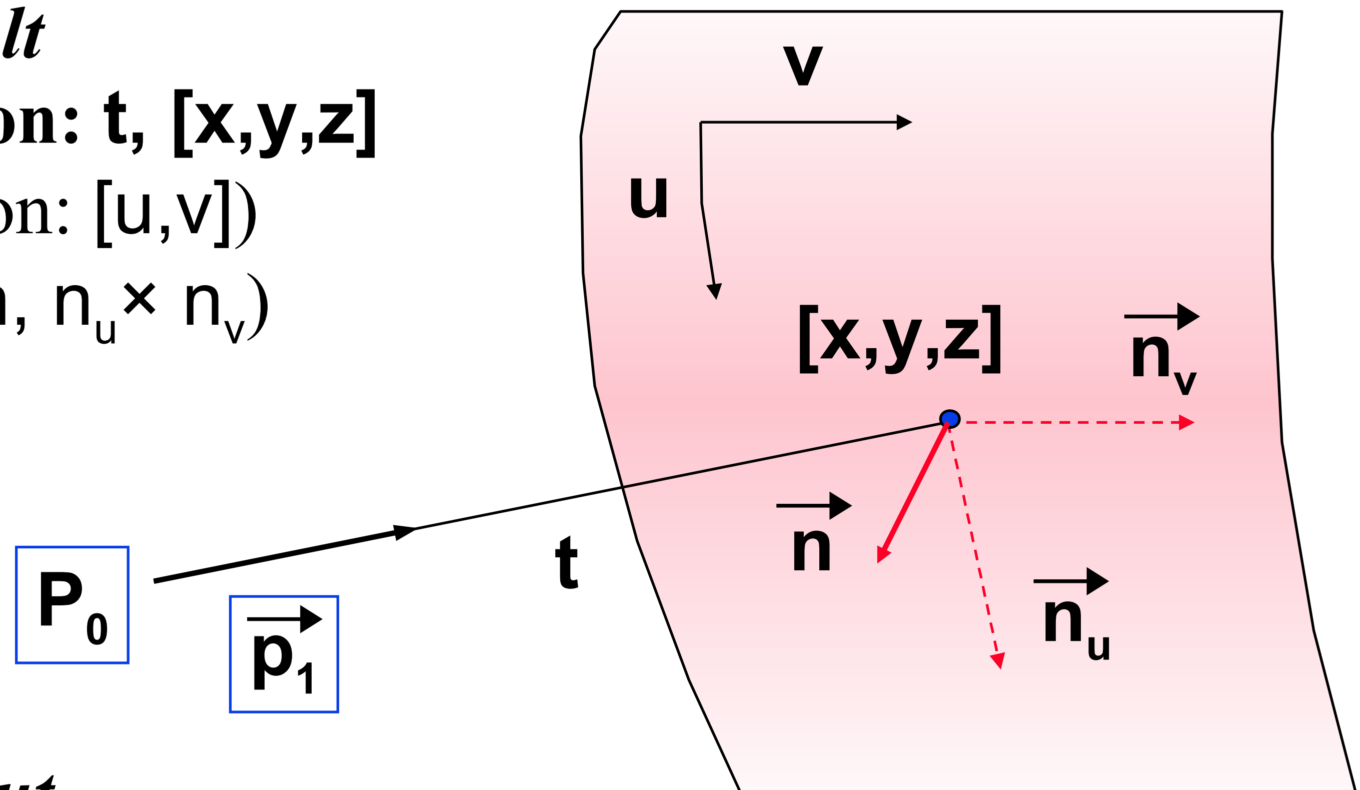
Ray × scene intersection

result

3D position: t, [x,y,z]

(2D position: [u,v])

(Normal: n, $n_u \times n_v$)



input

Ray: $P_0, \vec{p}_1$

Plane

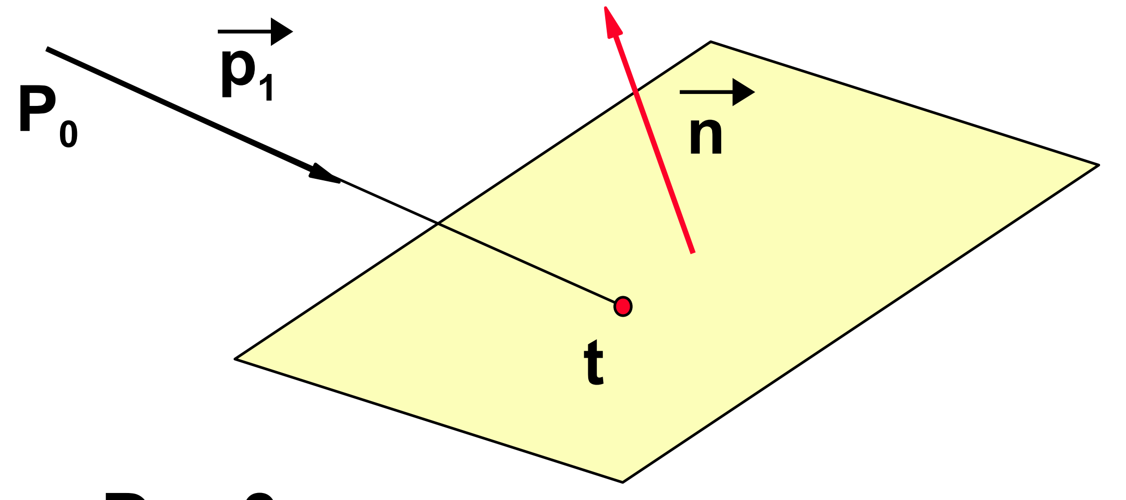
ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$

plane:

$$\vec{n} = [x_n, y_n, z_n]$$

$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$$



- intersection $\quad t = -(\vec{n} \cdot P_0 + D) \,/\, (\vec{n} \cdot \vec{p}_1)$
- negative: **2**±, **3***, positive: **5**±, **6***, **1**/
- computation of **[x,y,z]**: **3**±, **3***

Inverse transformation on the plane

plane:

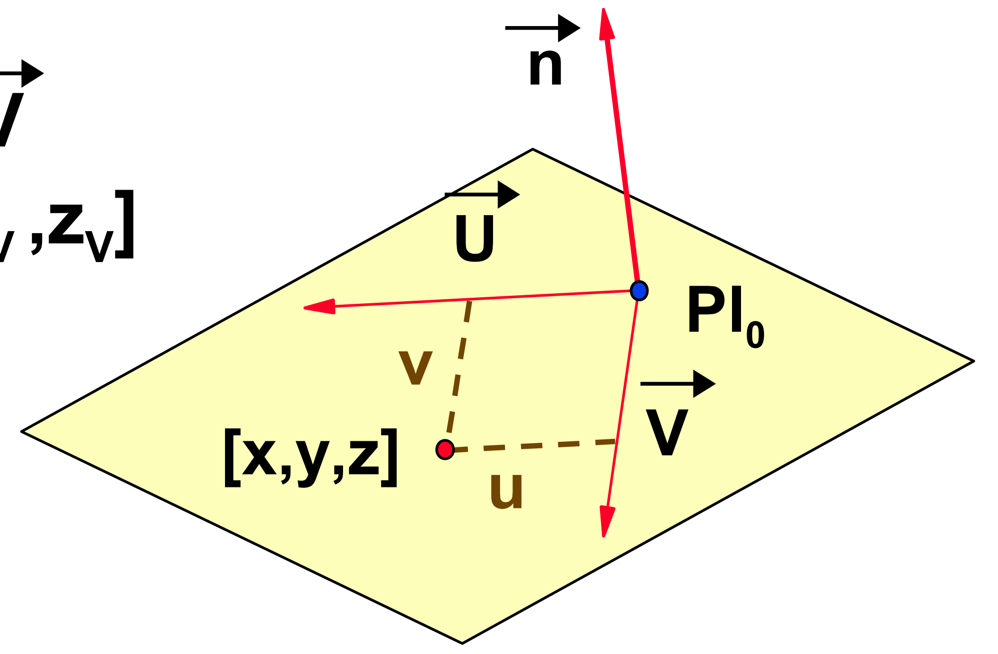
$$Pl(u,v) = Pl_0 + u \cdot \vec{U} + v \cdot \vec{V}$$

$$\vec{U} = [x_U, y_U, z_U],\ \vec{V} = [x_V, y_V, z_V]$$

$$\vec{n} = \vec{U} \times \vec{V}$$

input: $Pl, \vec{U}, \vec{V}, [x,y,z]$

result: $[u,v]$



- linear system $\underline{u} \cdot x_u + \underline{v} \cdot x_v = x - Pl_{0x}$

  $\underline{u} \cdot y_u + \underline{v} \cdot y_v = y - Pl_{0y}$

- solution **[u,v]**: **5±**, **5***, **2/**

Parallelogram

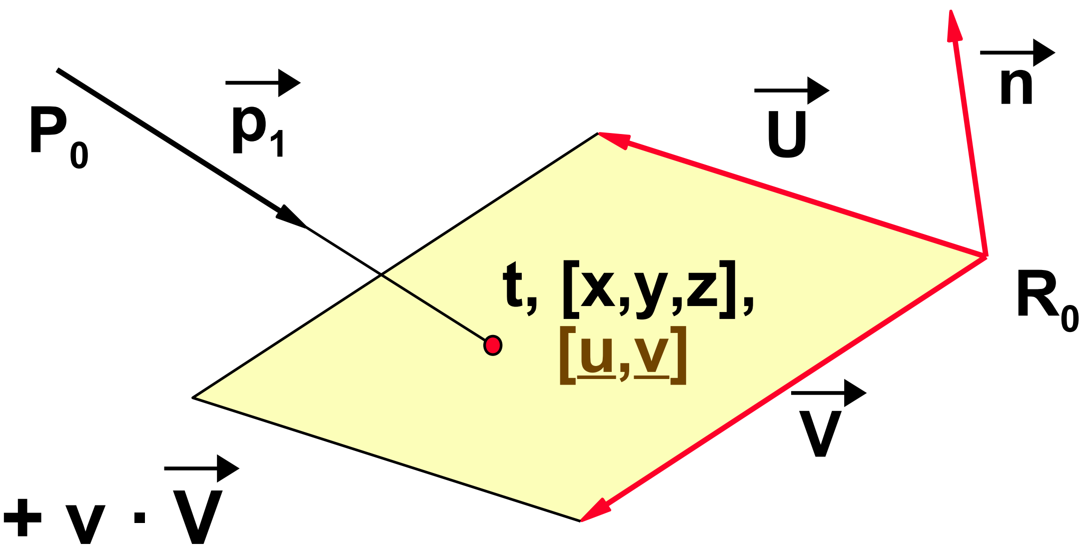
ray:

$P(t) = P_0 + t \cdot \vec{p_1}$

parallelogram:

$R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}$

$0 \le u,v \le 1$



- computing **t**, **[x,y,z]**, **[u,v]**, tests of **u,v**
- positive case total: **13±**, **14***, **3/**, **4≤**

Triangle

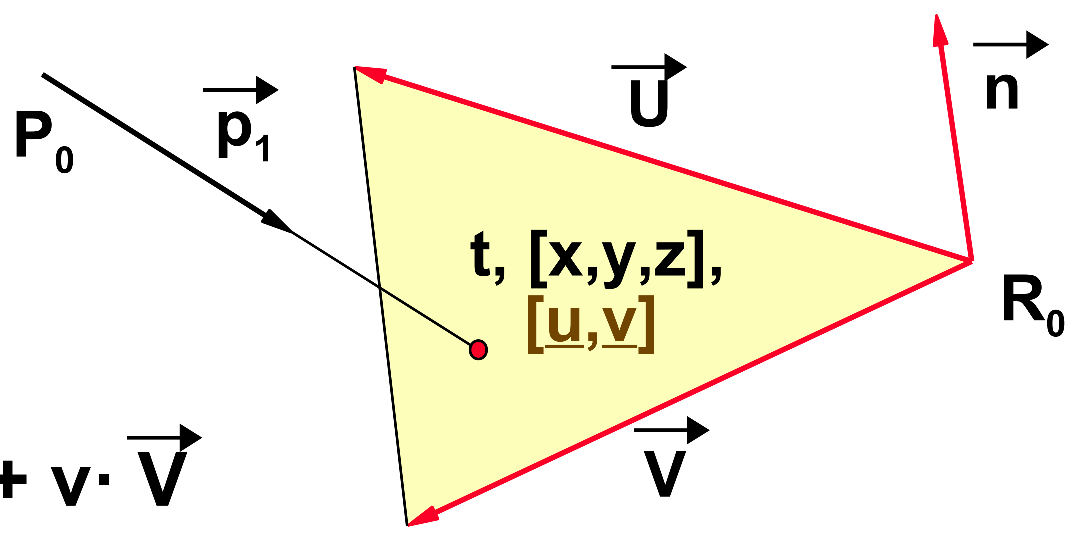
ray:

$$P(t) = P_0 + t \cdot \vec{p_1}$$

triangle:

$$R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}$$

$$0 \le u, v, \underline{u+v} \le 1$$



- computing **t**, **[x,y,z]**, **[u,v]**, tests of **u,v**
- positive case total: **14±**, **14***, **3/**, **3≤**

General planar polygon

ray:

$P(t) = P_0 + t \cdot \vec{p_1}$

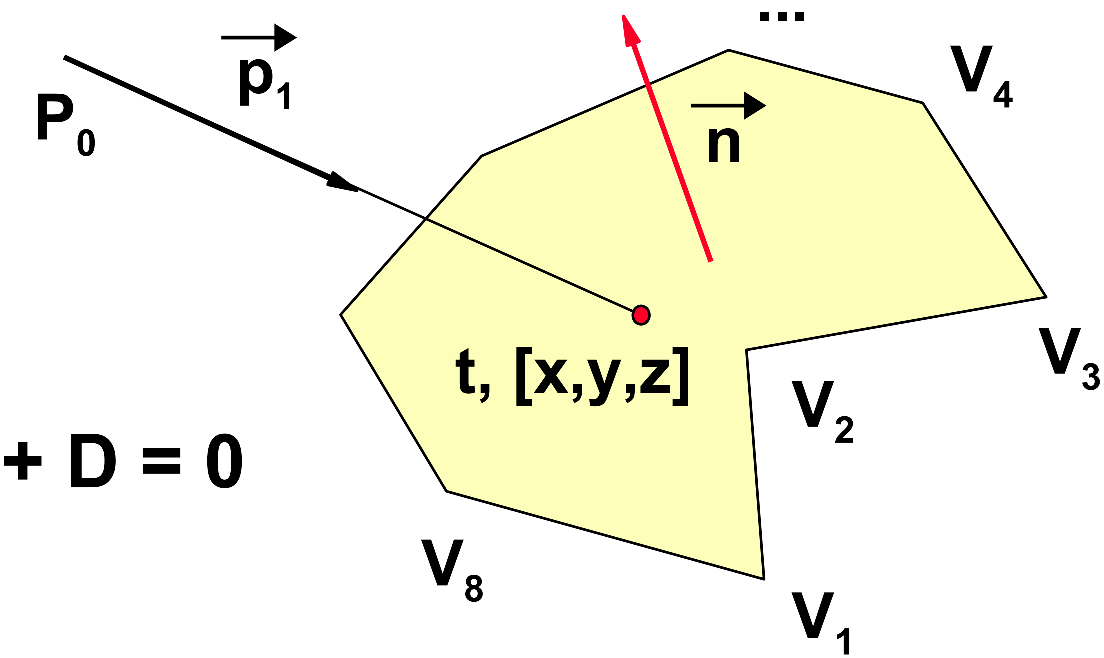
polygon plane:

$\vec{n} = [x_n, y_n, z_n]$

$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$

polygon vertices:

$V_1, V_2, \dots V_M$

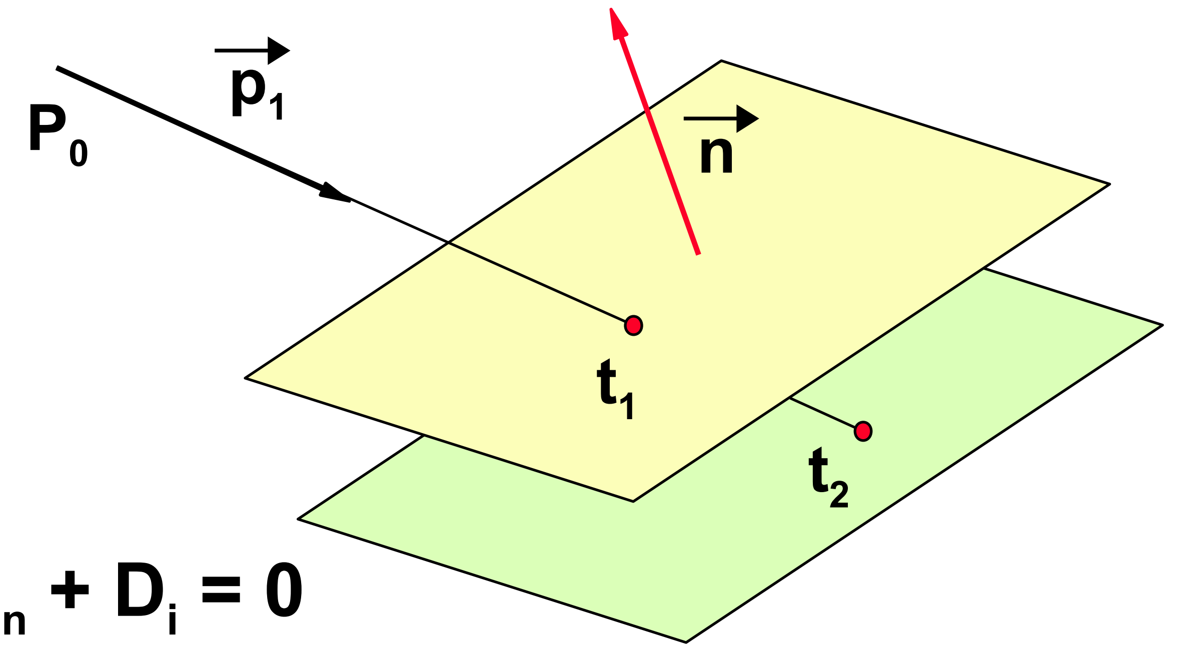


- computing **t**, **[x,y,z]**, planar test: **point × polygon**
- intersection with the plane: **8±, 9*, 1/**

Parallel planes

ray:

$$P(t) = P_0 + t \cdot \vec{p_1}$$



parallel planes:

$$\vec{n} = [x_n, y_n, z_n]$$

$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D_i = 0$$

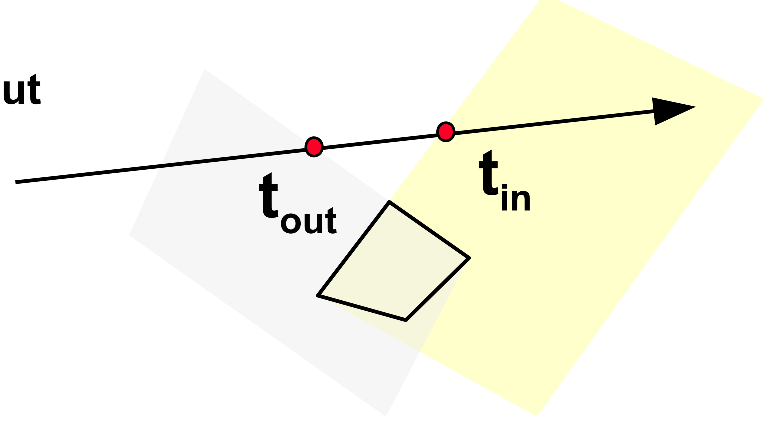
- intersections $t_i = -(\vec{n} \cdot P_0 + D_i) / (\vec{n} \cdot \vec{p_1})$
- the 1st plane: **5**±, **6***, **1**/, every next one: **1**±, **1**/

Convex polyhedron

- defined as an **intersection of K halfspaces**
  – at most **K** intersections ray vs. plane
  – **parallelism** of planes can be used – e.g. cuboid

- variables $\mathbf{t_{in}}, \mathbf{t_{out}}$ initialized to $\mathbf{0}, \infty$
- ray vs. one halfspace: $\langle \mathbf{t}, \infty)$ resp. $(-\infty, \mathbf{t} \rangle$

  $\mathbf{t_{in} = \max\{ t_{in} , t \}}$ resp. $\mathbf{t_{out} = \min\{ t_{out} , t \}}$
- early exit if $\mathbf{t_{in}} > \mathbf{t_{out}}$



Implicit surface

ray:

$$P(t) = P_0 + t \cdot \vec{p_1}$$

implicit surface:

$$F(x,y,z) = 0$$

example:

$$(c - \cos ax) \cos z + (y + a \sin ax) \sin z + \\ + \cos a(x+z) = 0$$

- substitution **P(t)** into **F**: **$F^*(t) = 0$**
- finding roots of the function **$F^*(t)$**
  - sometimes only the **smallest positive root** is needed (the 1st intersection), for **CSG** we need **all roots**

Algebraic surface

ray:

$$P(t) = P_0 + t \cdot \overrightarrow{p_1}$$

algebraic surface of degree **d**:

$$A(x, y, z) = \sum_{i,j,k=0}^{i+j+k \le d} a_{ijk} \cdot x^i y^j z^k = 0$$

example (toroid with radii **a, b**):

$$T_{ab}(x, y, z) = \left(x^2 + y^2 + z^2 - a^2 - b^2\right)^2 - 4a^2\left(b^2 - z^2\right)$$

- after substitution **P(t)** into **A**: **A*(t) = 0**
- **A*** is a polynomial of degree **d** (at most)

Quadric (d=2)

general quadric:

$$x^T Q x = 0 \qquad x = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

after substitution of P(t):

$$a_2 t^2 + a_1 t + a_0 = 0,$$

where $a_2 = P_1^T Q P_1, \quad a_1 = 2P_1^T Q P_0, \quad a_0 = P_0^T Q P_0$

Quadric of revolution

quadric of revolution in standard position:

$$x^2 + y^2 + az^2 + bz + c = 0$$

sphere:

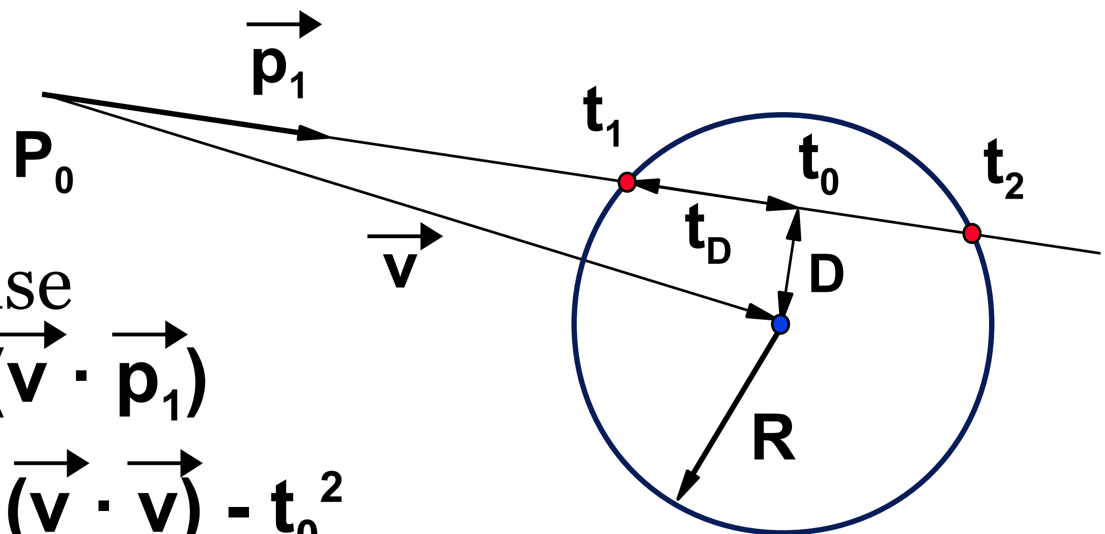
$$x^2 + y^2 + z^2 - 1 = 0,$$

after substitution of $P(t)$*:*

$$t^2(P_1 \cdot P_1) + 2t(P_0 \cdot P_1) + (P_0 \cdot P_0) - 1 = 0$$

Sphere (geometric solution)

$$P(t) = P_0 + t \cdot \vec{p_1}$$



- center of the subtense $t_0 = (\vec{v} \cdot \vec{p_1})$
- distance $D^2 = (\vec{v} \cdot \vec{v}) - t_0^2$
- inclination $t_D^2 = R^2 - D^2$
- for $t_D^2 = 0$ there is one tangent point $P(t_0)$
- for $t_D^2 > 0$ two intersections exist: $P(t_0 \pm t_D)$
- negative case: **9±**, **6***, **1<**, positive addit.: **2±**, **1 sqrt**

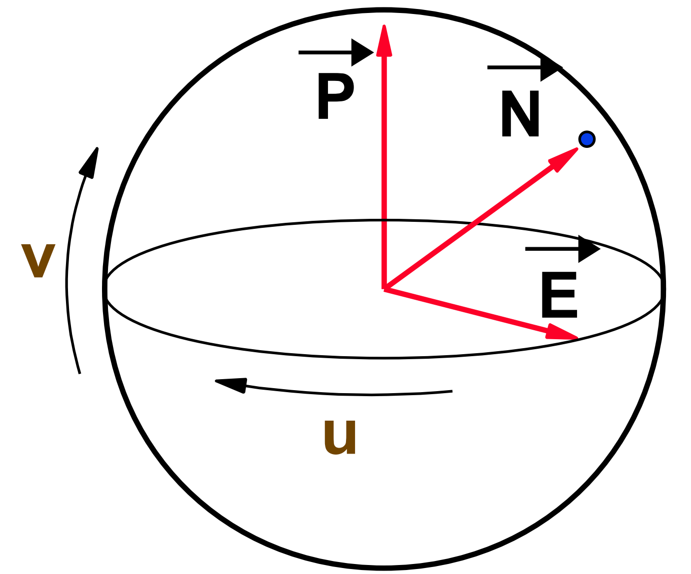
Inverse transformation on the sphere

sphere:

$$(x-x_c)^2+(y-y_c)^2+(z-z_c)^2 = R^2$$

pole dir: $\vec{P}$, *equator dir:* $\vec{E}$

$$(\vec{P} \cdot \vec{E}) = 0$$



input: $\vec{N}, \vec{P}, \vec{E}$

result: [u,v] from $[0,1]^2$

$$\Phi = \arccos\left(-\vec{N}\cdot\vec{P}\right), \quad \theta = \frac{\arccos\left[\left(\vec{N}\cdot\vec{E}\right)/\sin\Phi\right]}{2\pi}$$

$$v = \Phi/\pi, \quad \left(\vec{P}\times\vec{E}\right)\cdot\vec{N} > 0 \Rightarrow u = \theta, \quad \text{else} \quad u = 1-\theta$$

Cylinder and cone

unit cylinder and unit cone in basic position:

$$x^2 + y^2 - 1 = 0 \qquad x^2 + y^2 - z^2 = 0$$

after substitution* P(t) *for the cylinder:

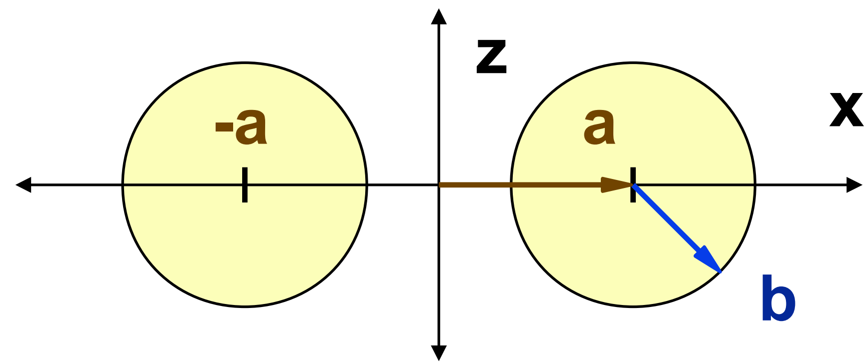
$$t^2\left(x_1^2 + y_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1\right) + x_0^2 + y_0^2 - 1 = 0$$

after substitution* P(t) *for the cone:

$$t^2\left(x_1^2 + y_1^2 - z_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1 - z_0 z_1\right) +$$

$$+ x_0^2 + y_0^2 - z_0^2 = 0$$

Toroid



Two circles in the **xz** *plane*:

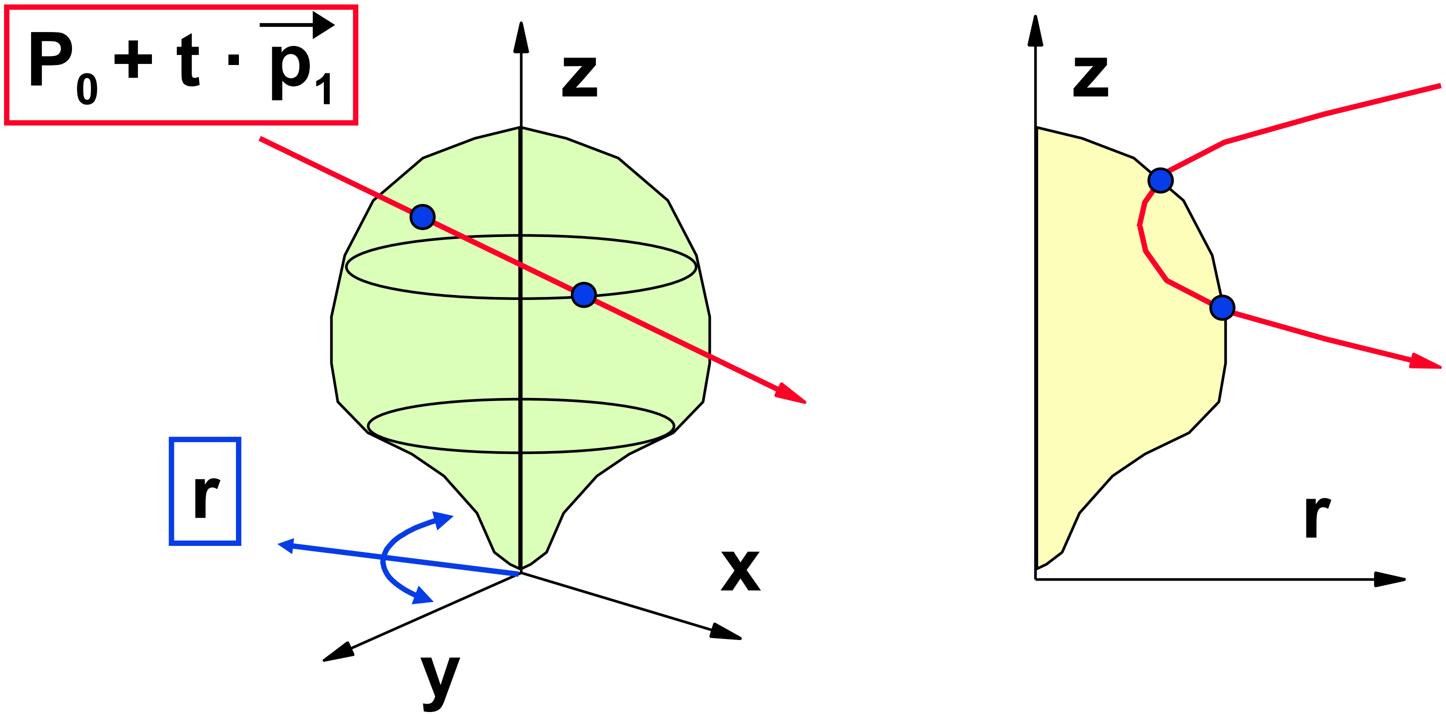
$$\left[(x-a)^2 + z^2 - b^2\right] \cdot \left[(x+a)^2 + z^2 - b^2\right] = 0$$

$$\left[x^2 + z^2 - \left(a^2 + b^2\right)\right]^2 = 4a^2\left(b^2 - z^2\right)$$

After substitution $r^2 = x^2 + y^2$ *for* x^2 *– the 4^th^ degree equation:*

$$\left(x^2 + y^2 + z^2 - a^2 - b^2\right)^2 - 4a^2\left(b^2 - z^2\right) = 0$$

Surface of revolution



equation of the ray in the **rz** *plane*:

$$r^2 = x^2 + y^2 = (x_0 + x_1 t)^2 + (y_0 + y_1 t)^2$$

$$z = z_0 + z_1 t$$

Ray in the rz plane

After elimination of **t:** $\mathbf{ar^2 + bz^2 + cz + d = 0}$ **(1)**

$$a = -z_1^{\ 2} \qquad e = x_0 x_1 + y_0 y_1$$

$$b = x_1^{\ 2} + y_1^{\ 2} \qquad f = x_0^{\ 2} + y_0^{\ 2}$$

$$c = 2(z_1 e - z_0 b)$$

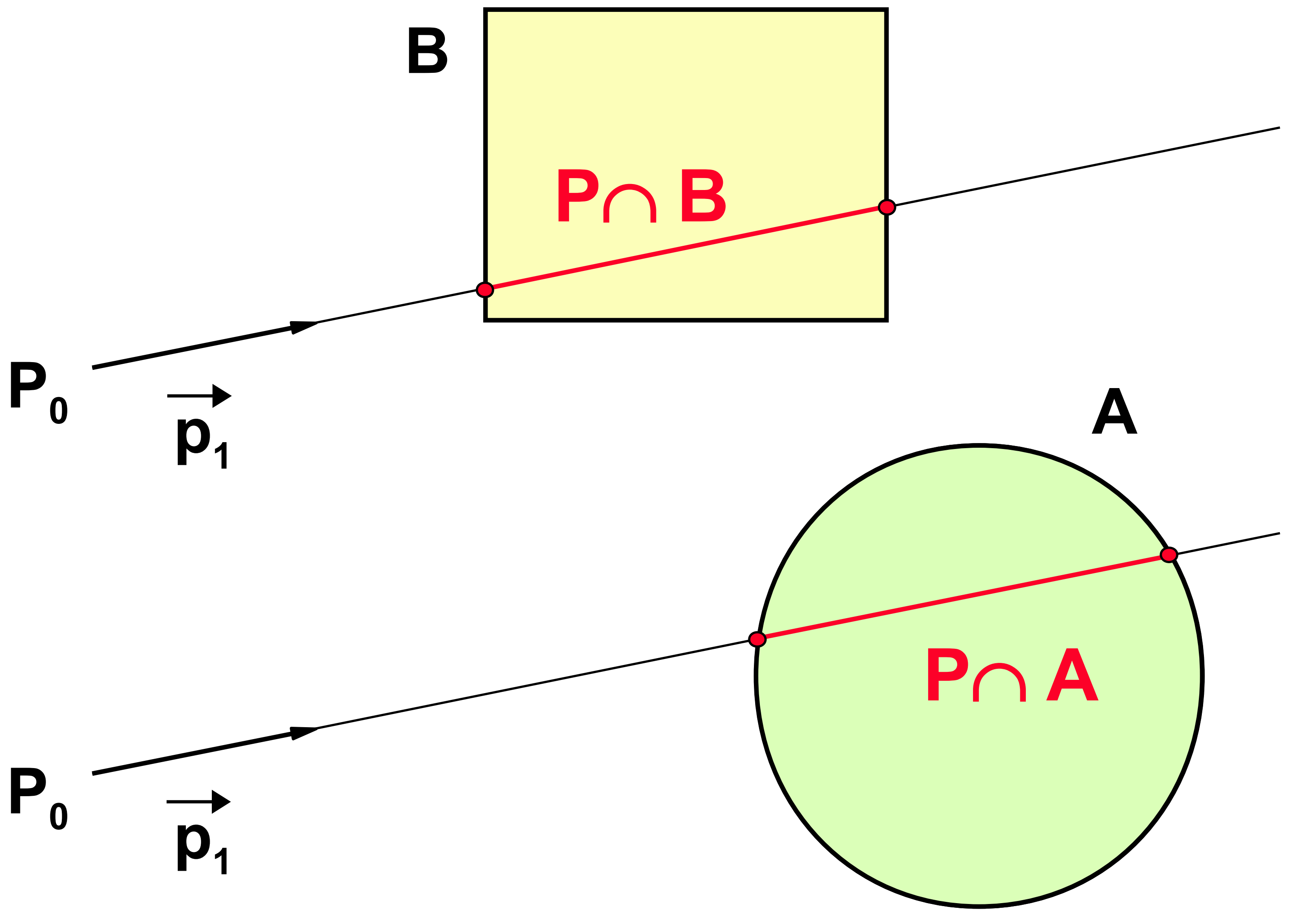
$$d = z_0(z_0 b - 2 z_1 e) + f z_1^{\ 2}$$

- after substitution of parametric curve **K(s)** into **(1)** we get an equation **K*(s) = 0**
- **K*** has got a double degree (compared to **K**)

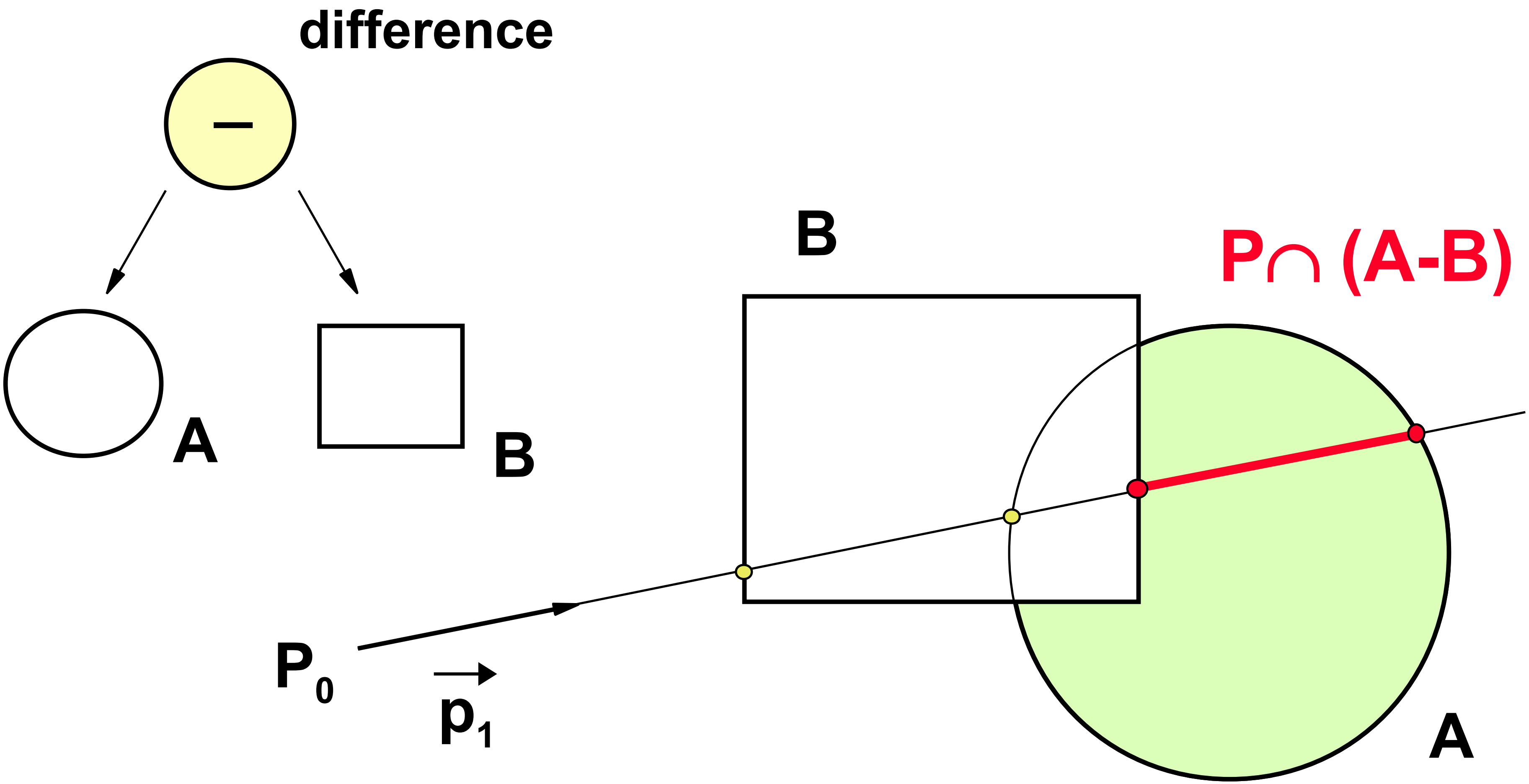
CSG representation

- **primitive solids** are easy
  - convex objects – only two intersections
- **set operations** are performed in the **1D ray-space**:
  - distributivity: $\mathbf{P \cap (A-B) = (P \cap A) - (P \cap B)}$
  - general ray-scene intersection is a collection of line segments (intervals in 1D ray-space)
- **geometric transformations**:
  - inverse transformation applied to a ray

Intersections P∩A, P∩B



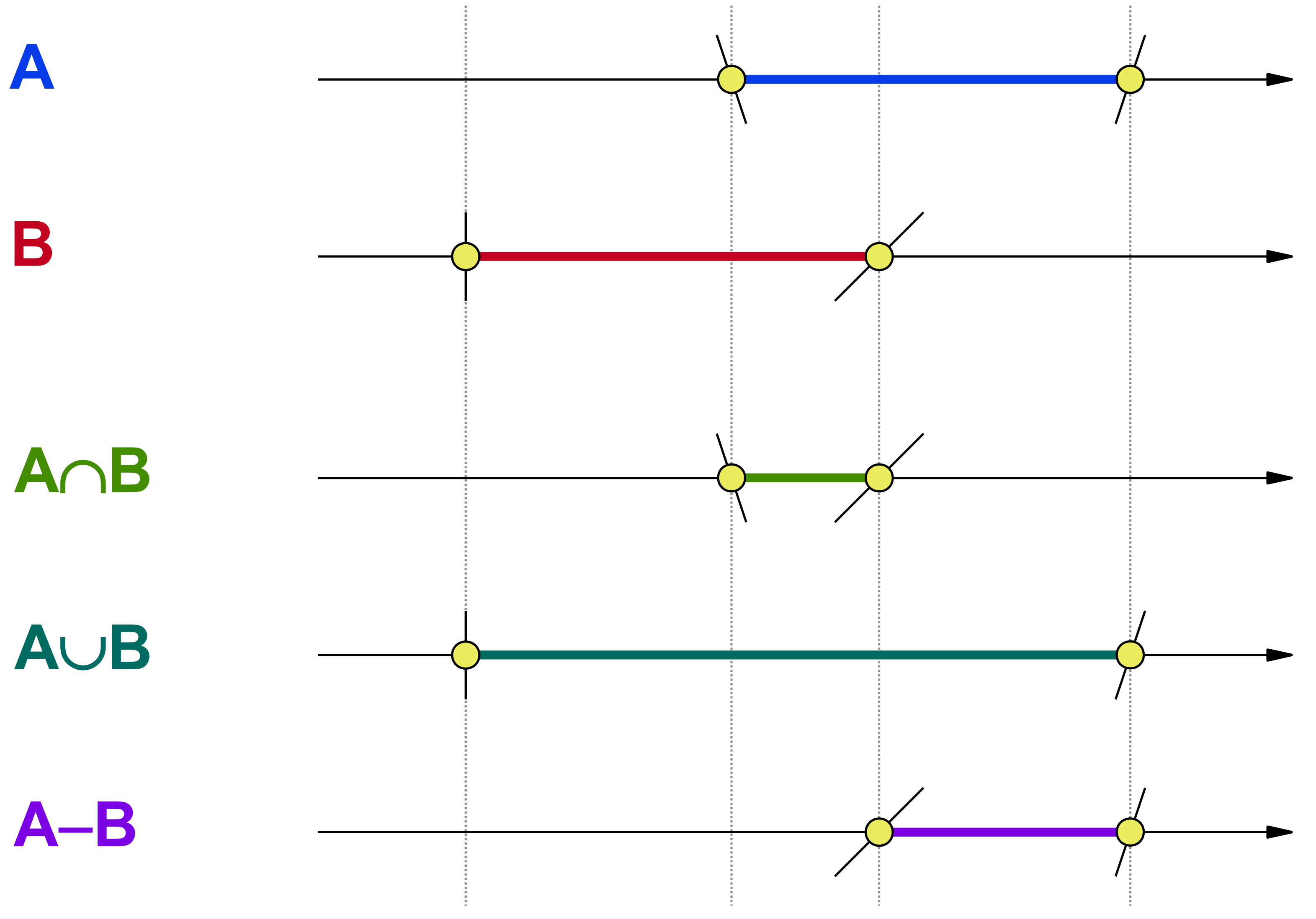
Intersection $P \cap (A-B)$



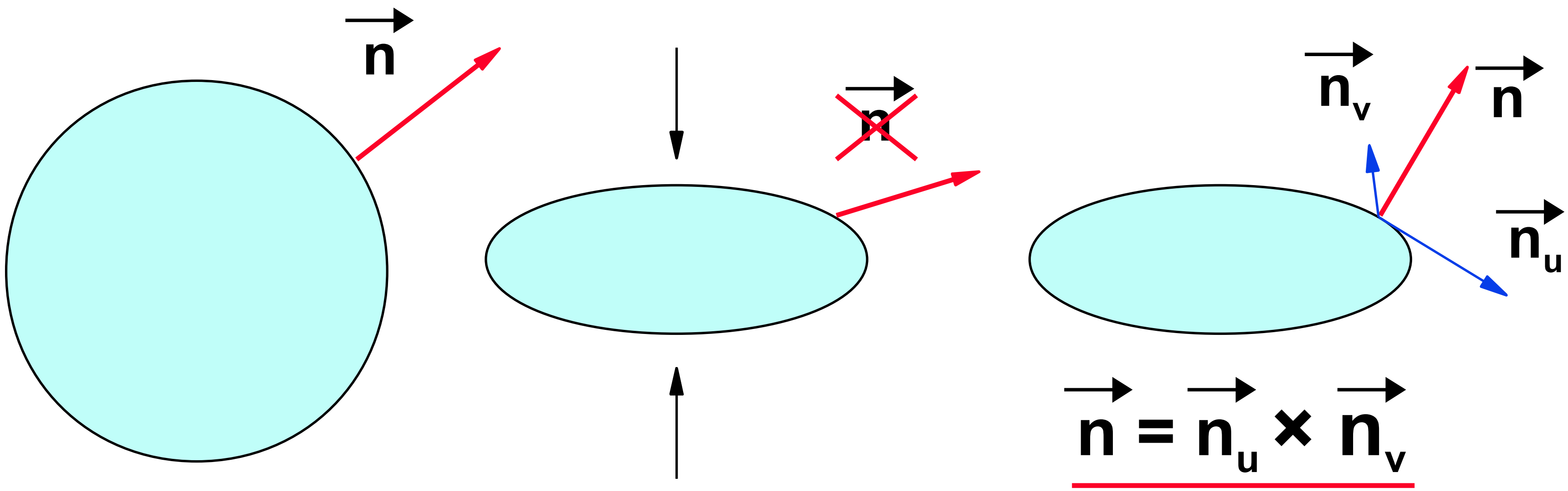
Implementation

- **ray**:
  - origin $\mathbf{P_0}$ and direction $\vec{\mathbf{p_1}}$
  - transforms with inverse matrices $\mathbf{T_i^{-1}}$ (could not be efficient enough … 1 transformation: **15+**, **18***)
- **ray vs. scene intersection** (partial & final):
  - ordered list of **t** parameter in ray-space: $[\mathbf{t_1}, \mathbf{t_2}, \mathbf{t_3}, ..]$
- **set operation**:
  - generalized merging of ordered lists $[\ \mathbf{t_i}\ ]$
- **transformation of normal vectors!**

Set operations on the ray



Normal vector transformation



- general **afine transformation doesn't keep angles**
  - **two tangent vectors** instead a normal
  - **tangent vectors** transformed by **3×3** submatrix only!
- alternative matrix for **normal vectors**: $M_n = (M^{-1})^T$

References

- **A. Glassner**: ***An Introduction to Ray Tracing***, Academic Press, London 1989, 35-119
- **J. Foley, A. van Dam, S. Feiner, J. Hughes**: ***Computer Graphics, Principles and Practice***, 712-714

