matfyz

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Ray vs. Scene Intersections

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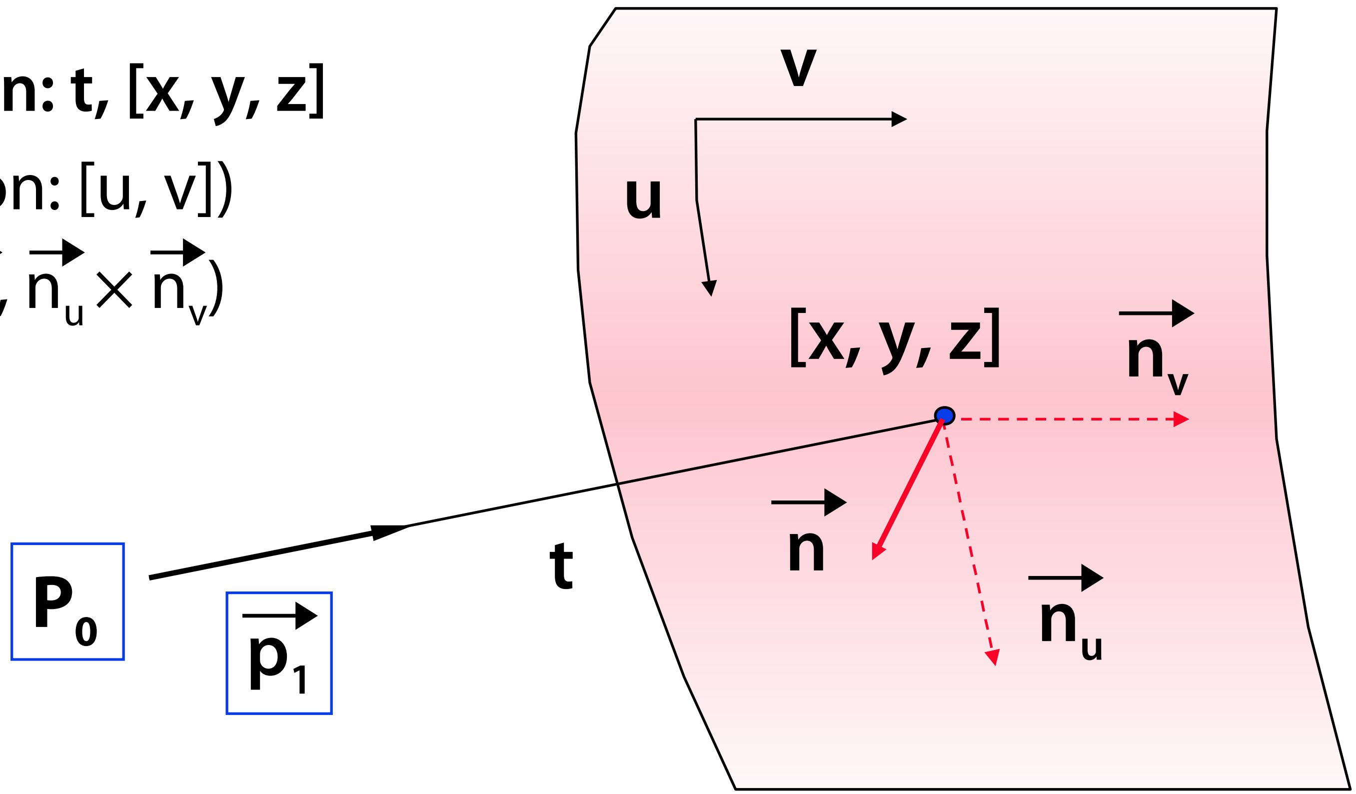
Ray vs. scene intersection

Result

3D position: t, [x, y, z]

(2D position: [u, v])

(Normal: $\vec{n}$, $\vec{n}_u \times \vec{n}_v$)



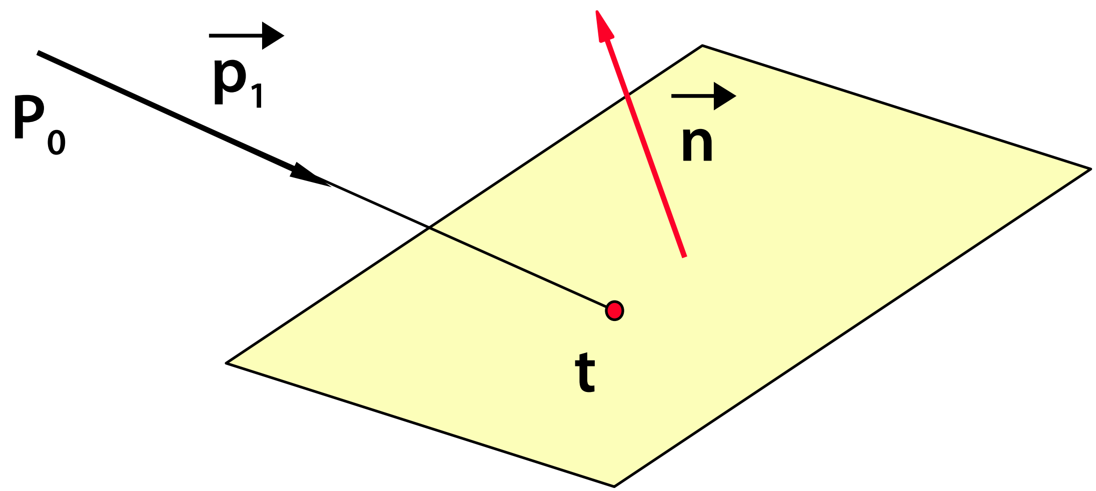
Input

Ray: P_0, $\vec{p}_1$

Plane

Ray

$P(t) = P_0 + t \cdot \vec{p_1}$



Plane

$\vec{n} = [x_n , y_n , z_n]$

$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$

Intersection $t = -(\vec{n} \cdot P_0 + D) / (\vec{n} \cdot \vec{p_1})$

Negative: **2±, 3***, positive: **5±, 6*, 1/**

Computation of **[x, y, z]: 3±, 3***

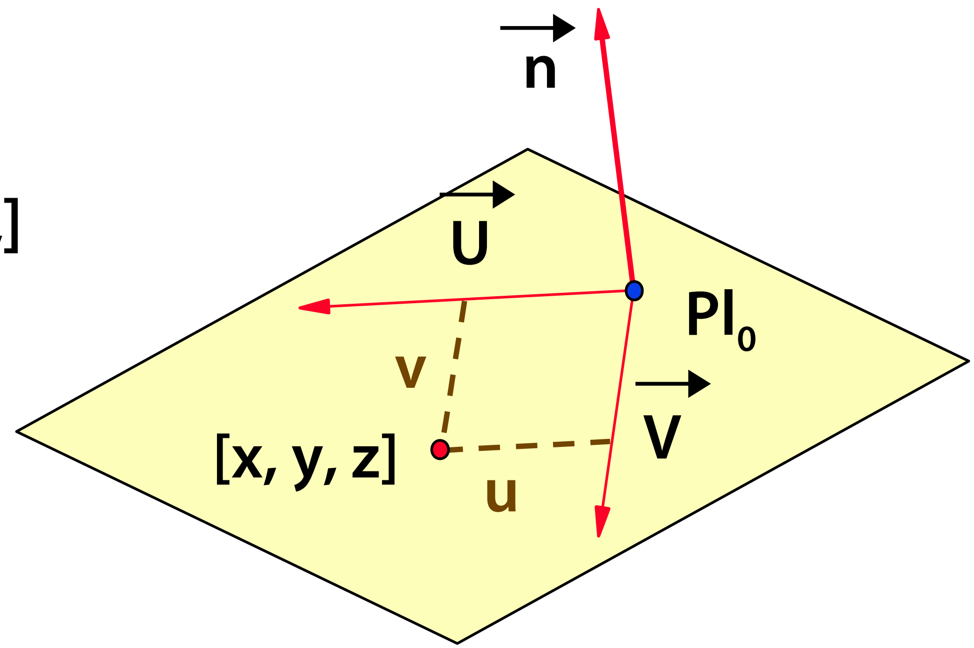
Inverse transformation on the plane

Plane

$$Pl(u,v) = Pl_0 + \vec{u} \cdot U + \vec{v} \cdot V$$

$$\vec{U} = [x_U, y_U, z_U], \vec{V} = [x_V, y_V, z_V]$$

$$\vec{n} = \vec{U} \times \vec{V}$$



Input $Pl_0, \vec{U}, \vec{V}, [x, y, z]$

Result **[u, v]**

Linear system

$$\underline{u} \cdot x_u + \underline{v} \cdot x_v = x - Pl_{0x}$$

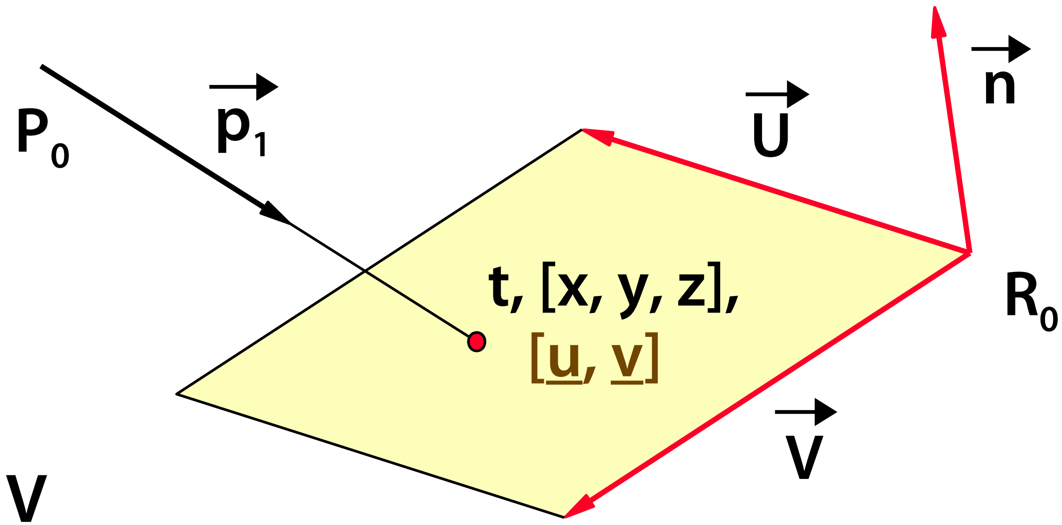
$$\underline{u} \cdot y_u + \underline{v} \cdot y_v = y - Pl_{0y}$$

Solution **[u, v]**: **5±**, **5***, **2/**

Parallelogram

Ray

$$P(t) = P_0 + t \cdot \vec{p}_1$$



Parallelogram

$$R(u,v) = R_0 + \vec{u} \cdot U + \vec{v} \cdot V$$

$$0 \leq u, v \leq 1$$

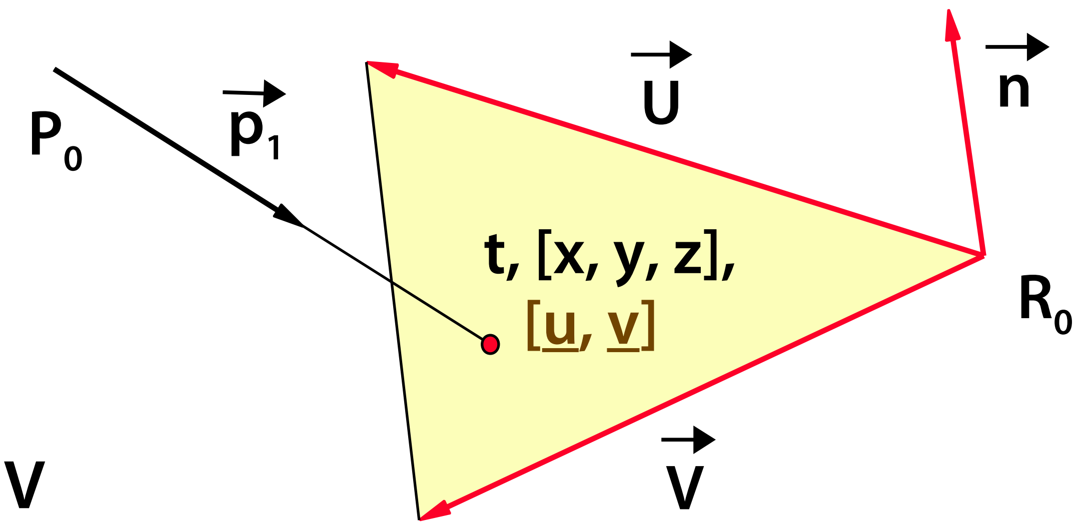
Computing **t, [x, y, z], [u, v],** tests of **u**, **v**

Positive case total **13±, 14*, 3/, 4≤**

Triangle

Ray

$$P(t) = P_0 + t \cdot \vec{p}_1$$



Triangle

$$R(u,v) = R_0 + \vec{u} \cdot U + \vec{v} \cdot V$$

$$0 \le u, v, \underline{u+v} \le 1$$

Computing **t, [x, y, z], [u, v],** tests of **u, v**

Positive case total **14±, 14*, 3/, 3≤**

General planar polygon

Ray

$\mathbf{P(t) = P_0 + t \cdot \vec{p}_1}$

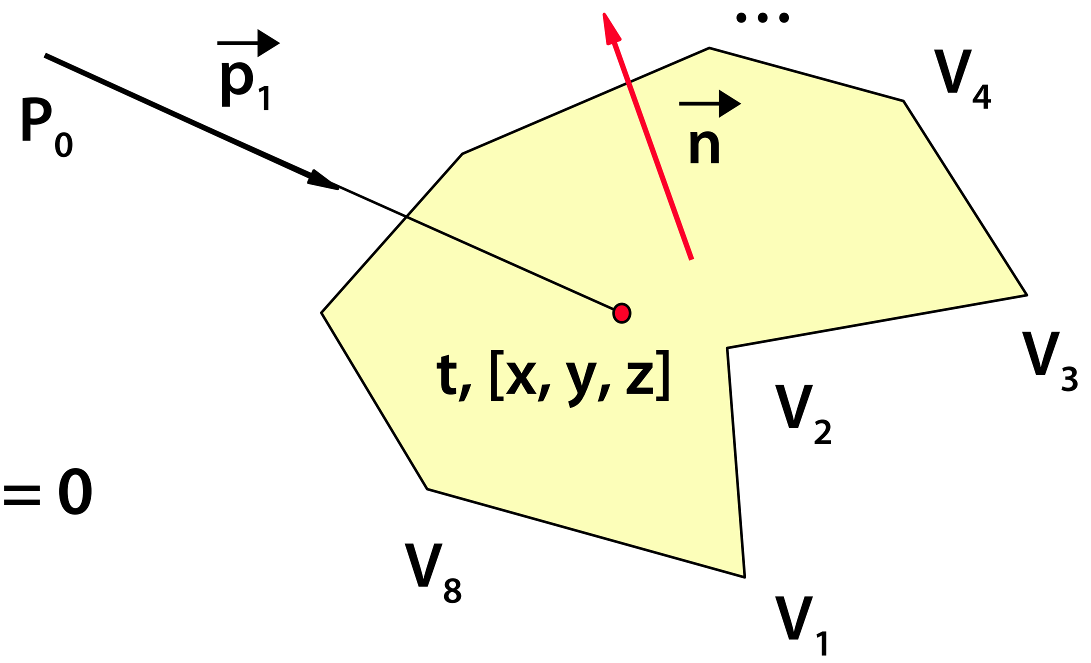
Polygon plane

$\mathbf{\vec{n} = [x_n, y_n, z_n]}$

$\mathbf{x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0}$

Polygon vertices

$\mathbf{V_1, V_2 \ldots V_M}$



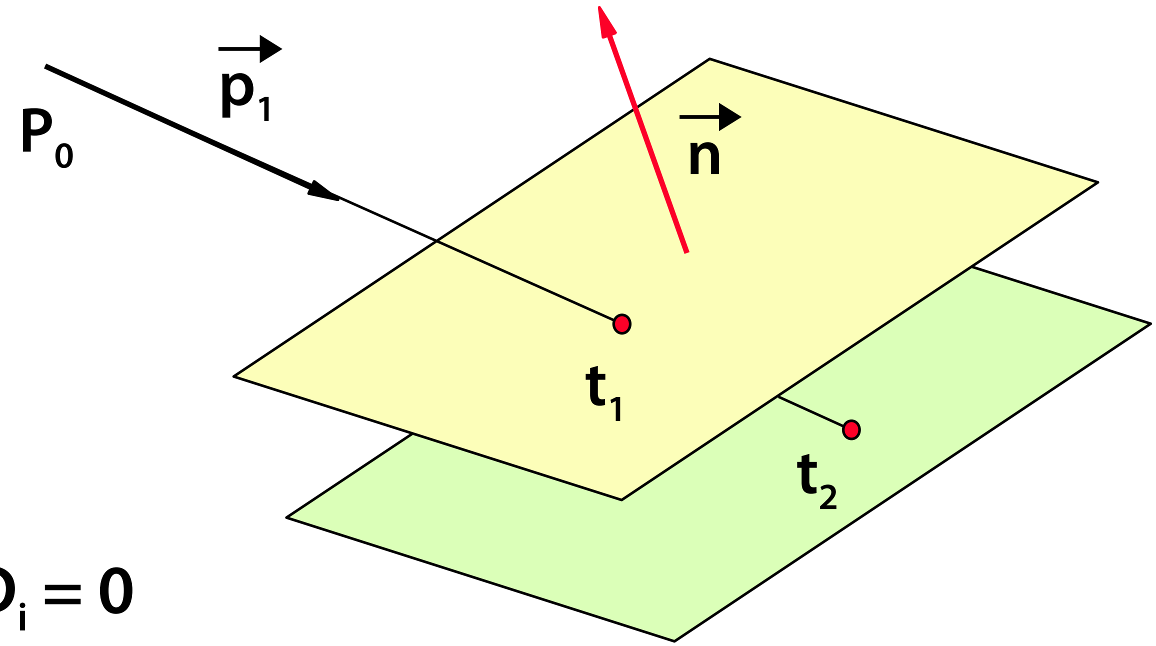
Computing **t, [x, y, z]**, planar test: **point × polygon**

Intersection with the plane: **8±, 9*, 1/**

Parallel planes

Ray

$\mathbf{P(t) = P_0 + t \cdot \vec{p}_1}$



Parallel planes

$\mathbf{\vec{n} = [x_n, y_n, z_n]}$

$\mathbf{x \cdot x_n + y \cdot y_n + z \cdot z_n + D_i = 0}$

Intersections $\mathbf{t_i = -(\vec{n} \cdot P_0 + D_i) / (\vec{n} \cdot \vec{p}_1)}$

The 1st plane **5±**, **6***, **1/**, every other plane: **1±**, **1/**

Convex polyhedron

Defined as an **intersection of K halfspaces**

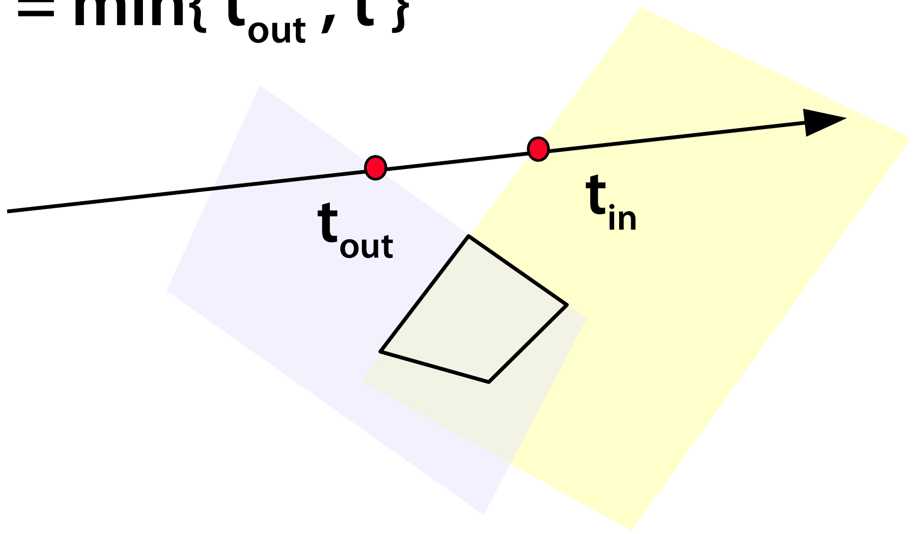
- at most **K** intersections ray vs. plane
- **parallelism** of planes can be used (cuboid)

Variables $\mathbf{t_{in}}, \mathbf{t_{out}}$ initialized to **0,** ∞

Ray vs. one halfspace: $\langle \mathbf{t}, \infty)$ resp. $(-\infty, \mathbf{t}\rangle$

$$\mathbf{t_{in} = max\{ t_{in} , t \}} \text{ resp. } \mathbf{t_{out} = min\{ t_{out} , t \}}$$

Early exit if $\mathbf{t_{in} > t_{out}}$



Implicit surface

Ray

$$\mathbf{P(t) = P_0 + t \cdot \vec{p}_1}$$

Implicit surface

$$\mathbf{F(x, y, z) = 0}$$

Example

$$\mathbf{(c - \cos ax) \cos z + (y + a \sin ax) \sin z + \cos a(x+z) = 0}$$

Substitution **P(t)** into **F** $\quad$ $\mathbf{F^*(t) = 0}$

Finding roots of the function **F*(t)**

- sometimes only the **smallest positive root** is needed (the 1st intersection), for **CSG** we will need **all roots**

Algebraic surface

Ray

$$P(t) = P_0 + t \cdot \vec{p}_1$$

Algebraic surface of degree **d**

$$A(x,y,z) = \sum_{i,j,k=0}^{i+j+k \le d} a_{ijk} \cdot x^i y^j z^k = 0$$

Example (toroid with radii a, b)

$$T_{ab}(x,y,z) = \left(x^2 + y^2 + z^2 - a^2 - b^2\right)^2 - 4a^2\left(b^2 - z^2\right)$$

After substitution **P(t)** into **A**: $A^*(t) = 0$

A* is a polynomial of degree **d** (at most)

Quadric (d=2)

General quadric

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = 0 \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

After substitution of **P(t)**

$$\mathbf{a}_2\mathbf{t}^2 + \mathbf{a}_1\mathbf{t} + \mathbf{a}_0 = 0,$$

where $\mathbf{a}_2 = \mathbf{P}_1^T\mathbf{Q}\mathbf{P}_1, \quad \mathbf{a}_1 = 2\mathbf{P}_1^T\mathbf{Q}\mathbf{P}_0, \quad \mathbf{a}_0 = \mathbf{P}_0^T\mathbf{Q}\mathbf{P}_0$

Quadric of revolution

Quadric of revolution in standard position

$$\mathbf{x^2 + y^2 + az^2 + bz + c = 0}$$

Sphere

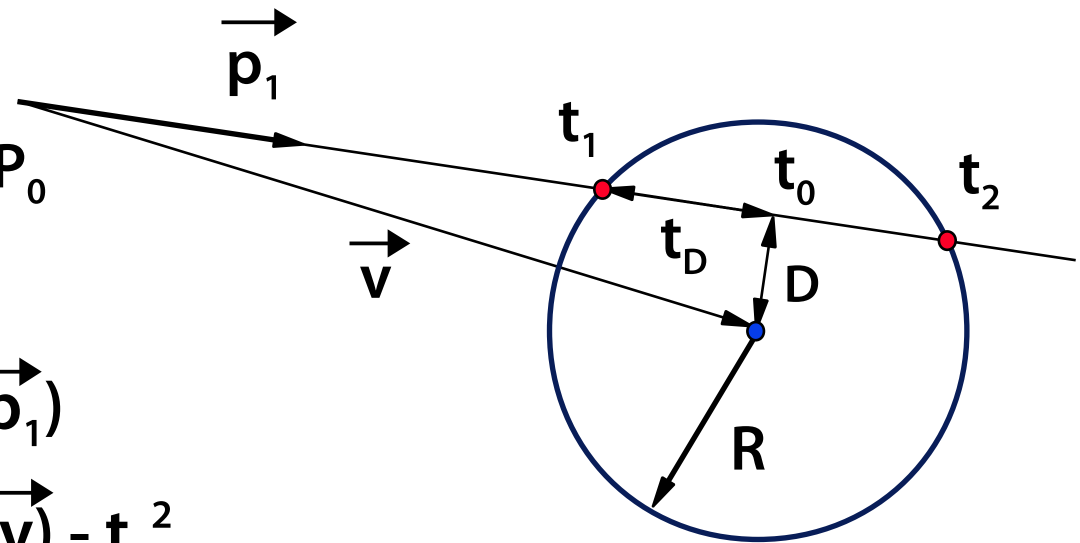
$$\mathbf{x^2 + y^2 + z^2 - 1 = 0,}$$

After substitution of **P(t)**

$$\mathbf{t^2(P_1 \cdot P_1) + 2t(P_0 \cdot P_1) + (P_0 \cdot P_0) - 1 = 0}$$

Sphere (geometric solution)

$$P(t) = P_0 + t \cdot \vec{p}_1$$



Center of the subtense

$$t_0 = (\vec{v} \cdot \vec{p}_1)$$

Distance $\quad D^2 = (\vec{v} \cdot \vec{v}) - t_0^{\,2}$

Inclination $\quad t_D^{\,2} = R^2 - D^2$

For $t_D^{\,2} = 0$ there is one tangent point $P(t_0)$

For $t_D^{\,2} > 0$ two intersections exist: $P(t_0 \pm t_D)$

Negative case: **9±, 6*, 1<**, positive – additonal: **2±, 1 sqrt**

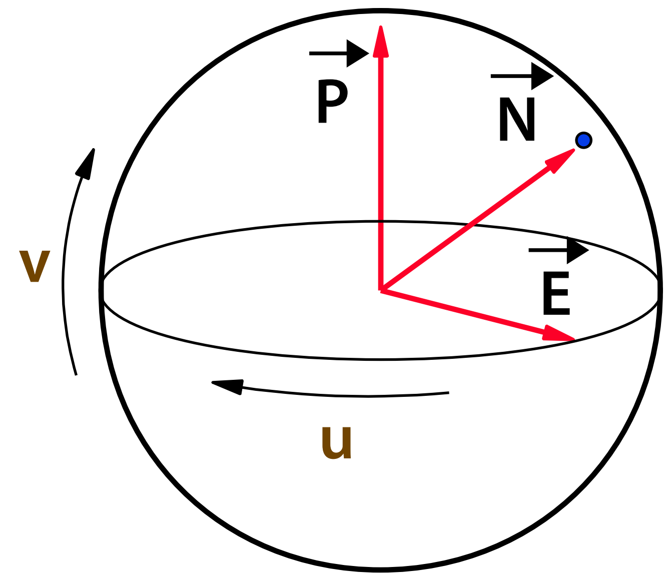
Inverse transformation on the sphere

Sphere

$$(x-x_c)^2+(y-y_c)^2+(z-z_c)^2 = R^2$$

Pole dir: $\vec{P}$, equator dir: $\vec{E}$

$$(\vec{P} \cdot \vec{E}) = 0$$



Input $\vec{N}, \vec{P}, \vec{E}$

Result [u, v] from $[0,1]^2$

$$\Phi = \arccos\left(-\vec{N} \cdot \vec{P}\right), \quad \theta = \frac{\arccos\left[\left(\vec{N} \cdot \vec{E}\right) / \sin\Phi\right]}{2\pi}$$

$$v = \Phi / \pi, \quad \left(\vec{P} \times \vec{E}\right) \cdot \vec{N} > 0 \Rightarrow u = \theta, \quad \text{else} \quad u = 1 - \theta$$

Cylinder and cone

Unit cylinder and unit cone in canonic position

$$x^2 + y^2 - 1 = 0 \qquad x^2 + y^2 - z^2 = 0$$

After substitution **P(t)** for the cylinder

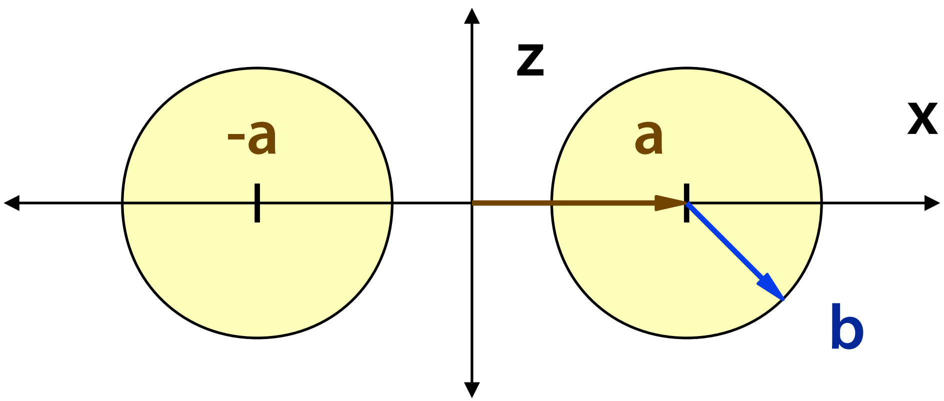
$$t^2\left(x_1^2 + y_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1\right) + x_0^2 + y_0^2 - 1 = 0$$

After substitution **P(t)** for the cone

$$t^2\left(x_1^2 + y_1^2 - z_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1 - z_0 z_1\right) +$$

$$+ x_0^2 + y_0^2 - z_0^2 = 0$$

Toroid



Two circles in the xz plane

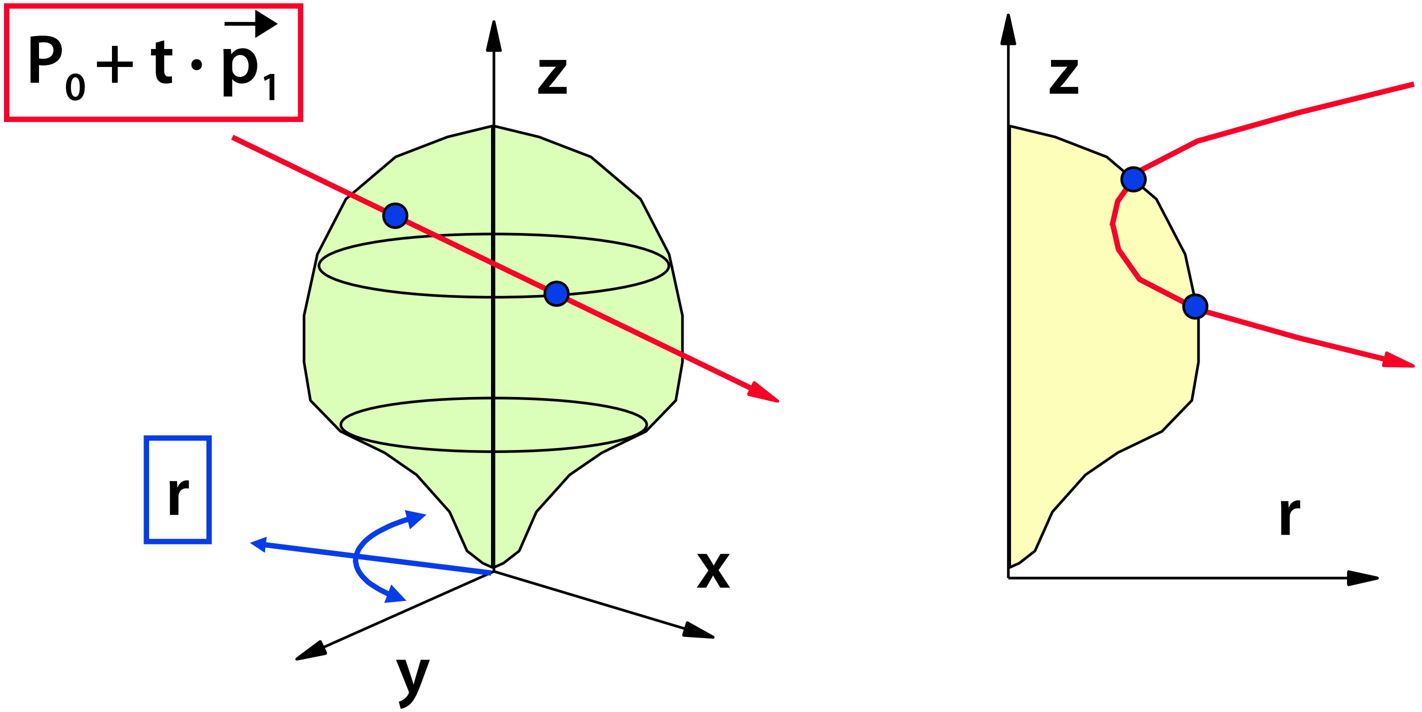
$$\left[(x-a)^2 + z^2 - b^2\right] \cdot \left[(x+a)^2 + z^2 - b^2\right] = 0$$

$$\left[x^2 + z^2 - (a^2 + b^2)\right]^2 = 4a^2(b^2 - z^2)$$

After substitution $r^2 = x^2 + y^2$ for x^2 – the 4th degree equation

$$\left(x^2 + y^2 + z^2 - a^2 - b^2\right)^2 - 4a^2(b^2 - z^2) = 0$$

Surface of revolution



Equation of the ray in the **rz** plane

$$r^2 = x^2 + y^2 = \left(x_0 + x_1 t\right)^2 + \left(y_0 + y_1 t\right)^2$$

$$z = z_0 + z_1 t$$

Ray in the rz plane

After elimination of **t** $\quad \mathbf{ar^2 + bz^2 + cz + d = 0} \qquad$ **(1)**

$$a = -z_1^2 \qquad e = x_0 x_1 + y_0 y_1$$

$$b = x_1^2 + y_1^2 \qquad f = x_0^2 + y_0^2$$

$$c = 2(z_1 e - z_0 b)$$

$$d = z_0(z_0 b - 2 z_1 e) + f z_1^2$$

After substitution of parametric curve **K(s)** into **(1)** we get an equation **K*(s) = 0**

K* has got a double degree (compared to **K**)

CSG representation

Primitive solids are easy

- convex objects – only two intersections

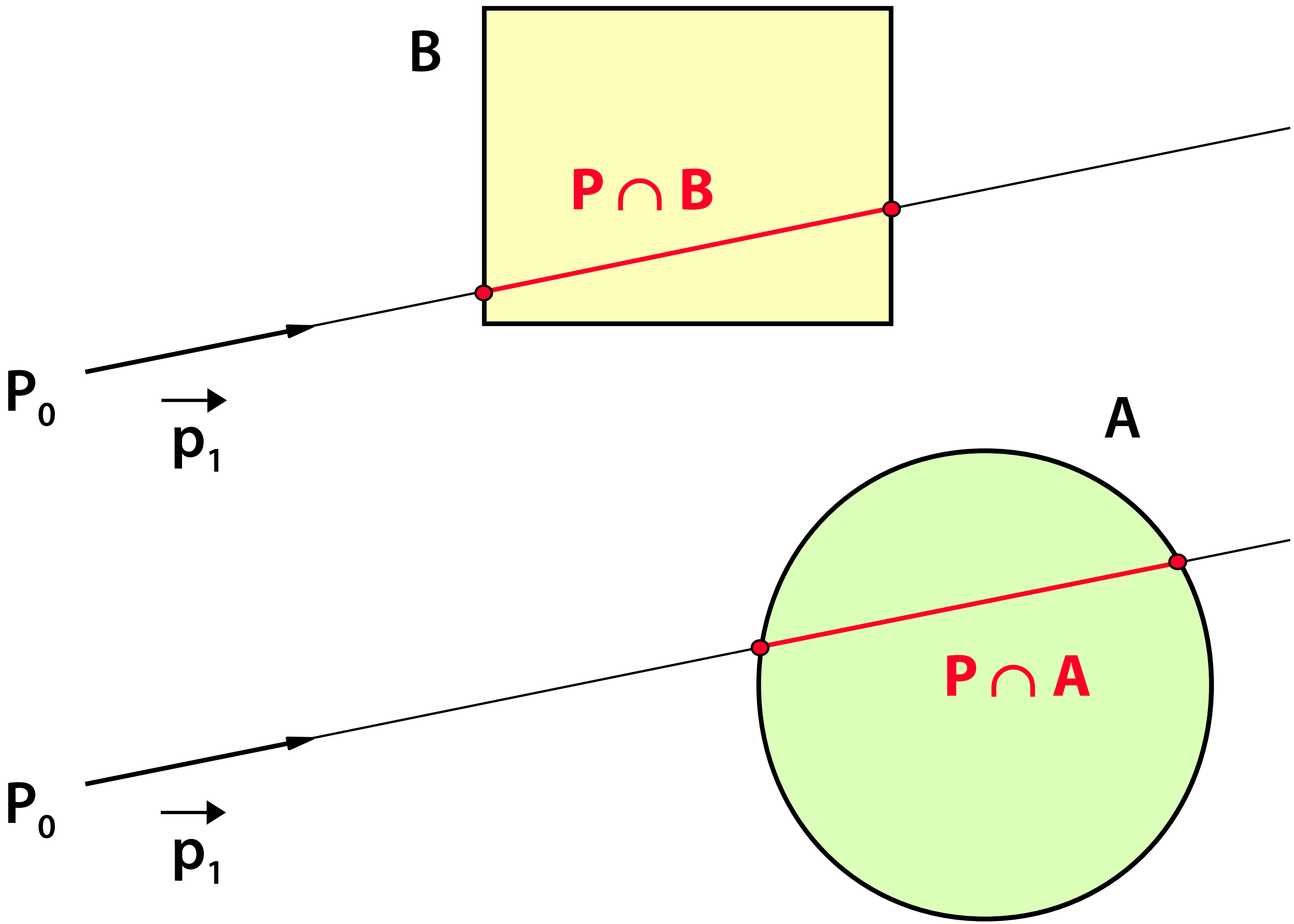
Set operations are performed in the **1D ray-space**

- distributivity: $\mathbf{P \cap (A - B) = (P \cap A) - (P \cap B)}$
- general ray-scene intersection is a collection of line segments (intervals in 1D ray-space)

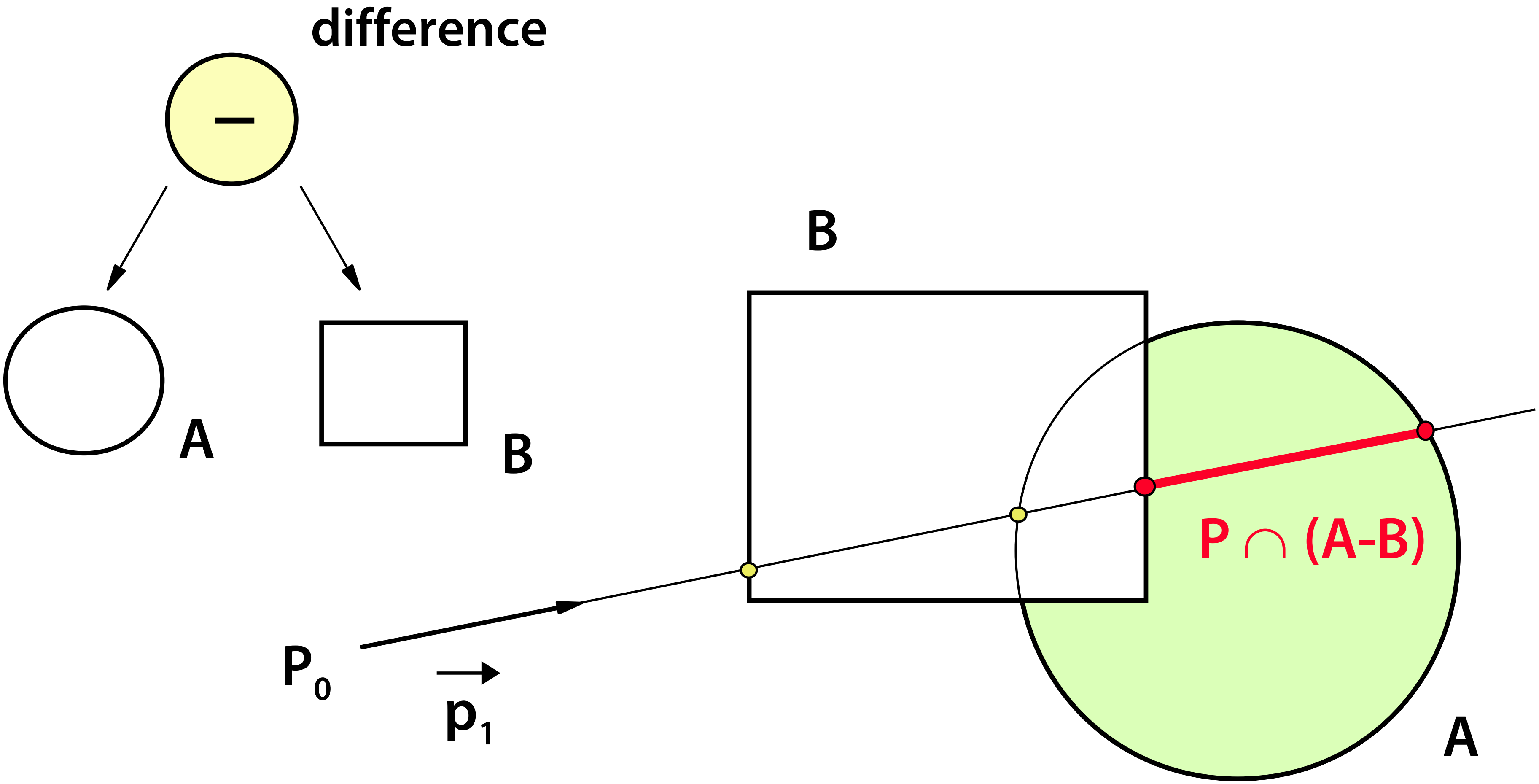
Geometric transformations

- inverse transformation applied to a ray

Intersections P∩ A, P∩ B



Intersection P∩ (A-B)



Implementation

Ray

- origin $\mathbf{P_0}$ and direction $\vec{\mathbf{p}}_1$
- transforms with inverse matrices $\mathbf{T}_i^{-1}$ (could not be efficient enough … 1 transformation: **15+**, **18***)

Ray vs. scene intersection (partial & final)

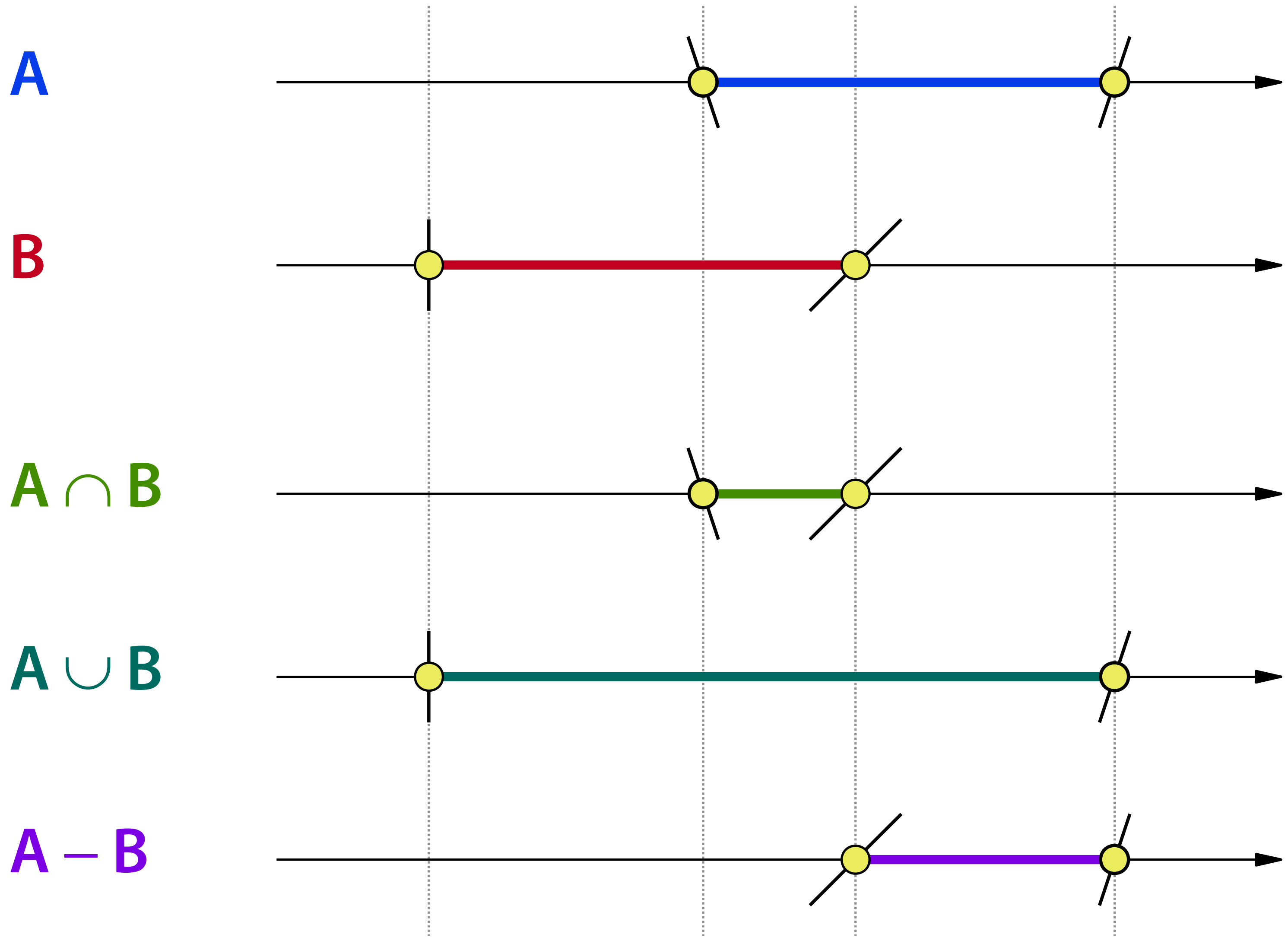
- ordered list of **t** parameter in ray-space **[**$\mathbf{t_1}$**,** $\mathbf{t_2}$**,** $\mathbf{t_3}$**…]**

Set operation

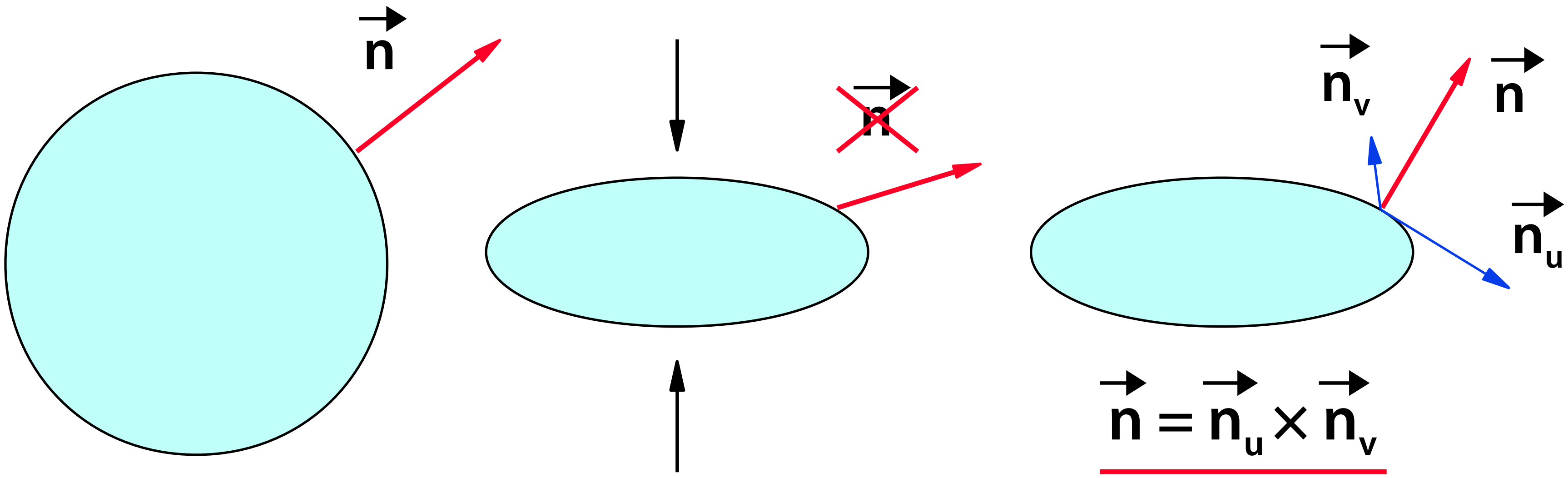
- generalized merging of ordered lists **[** $\mathbf{t_i}$ **]**

Transformation of normal vectors!

Set operations on the ray



Normal vector transformation



General **afine transformation doesn't preserve angles**

- **two tangent vectors** instead a normal
- **tangent vectors** transformed by **3×3** submatrix only!

Alternative matrix for **normal vectors** $M_n = (M^{-1})^T$

Literature

A. Glassner: ***An Introduction to Ray Tracing***, Academic Press, London 1989, 35-119

J. Foley, A. van Dam, S. Feiner, J. Hughes: ***Computer Graphics, Principles and Practice***, 712-714