Path Integral Methods for Light Transport Simulation: Theory & Practice

Introduction to Markov Chain and Sequential Monte Carlo
Markov Chains
Markov Chain

- Random walk implies a *transition probability* for each move
  \[ P(x_{n+1} = j|x_n = i) \equiv P_{i \rightarrow j} \]
- At each move the chain forms a *posterior distribution* over state space
  - A histogram of all visited states up to move \( n \)
- *Detailed balance* defined as \( P_{i \rightarrow j} = P_{j \rightarrow i} \)
Markov Chain

• Posterior converges to the *target* distribution *if* the detailed balance obeyed and all states are reachable (*ergodicity*)

• With “bad” initial state $x_0$ the *start-up bias* (burn-in phase) can be significant
Metropolis-Hastings Algorithm
Metropolis-Hastings (MH) Algorithm

- Goal: Random walk according to a desired function $f$
- Define conditional rejection sampling probability

$$a_{i\rightarrow j} = \frac{f(x_j)}{f(x_i)} = \frac{f_j}{f_i}$$

- $a_{i\rightarrow j}$ is acceptance probability at state $i$ for proposal state $j$
- Detailed balance is affected as $a_{i\rightarrow j}P_{i\rightarrow j} = a_{j\rightarrow i}P_{j\rightarrow i}$
- Posterior distribution is then proportional to $f$

- Accurate to a scaling factor = normalization constant
Metropolis-Hastings: Example
Metropolis-Hastings: Example

\[ a_{x_0 \rightarrow x_1} = \frac{N(x_1)}{N(x_0)} > 1 \]
Metropolis-Hastings: Example

\[ a_{x_1 \rightarrow x_2} = \frac{N(x_2)}{N(x_1)} \ll 1 \]
Metropolis-Hastings: Example

\[ x_2 \quad x_3 \]
Metropolis-Hastings: Example

$x_4'$

$x_3$
Metropolis-Hastings: Example

$x_4$

$x_5'$
Metropolis-Hastings: Example

\[ n = 20 \]
Metropolis-Hastings: Example

\[ n = 200 \]
Metropolis-Hastings: Example

\[ n = 2000 \]
Importance Sampling for M-H

- Cannot fetch proposals directly from $f$
- Generate a proposal $j$ from some proposal distribution $T$
  - Similar to importance sampling in Monte Carlo
  - $T$ can depend on the current state $i: T_{i\rightarrow j}$
  - New transition probability $P_{i\rightarrow j} = a_{i\rightarrow j}T_{i\rightarrow j}$
- Acceptance probability is then (from detailed balance):

$$a_{i\rightarrow j} = \left( \frac{f_j}{T_{i\rightarrow j}} \right) / \left( \frac{f_i}{T_{j\rightarrow i}} \right)$$
### Correspondence Table

<table>
<thead>
<tr>
<th>Ordinary Monte Carlo</th>
<th>Markov chain Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convergence rate, usually $O\left(\frac{1}{\sqrt{N}}\right)$</td>
<td>Mixing rate, depends on multiple factors, can be geometric $O(\gamma^N)$, $\gamma \in (0; 1)$</td>
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<td>Convergence to an expected value</td>
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<td>Importance sampling distribution $p(x)$</td>
<td>Proposal distribution $T_{i \rightarrow j}$</td>
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<td>Variance of the estimate</td>
<td>Acceptance rate, correlation of samples</td>
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<td>Number of moves (mutations)</td>
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Corresponding formula:

$\text{Convergence rate, usually } O\left(\frac{1}{\sqrt{N}}\right)$

$\text{Mixing rate, depends on multiple factors, can be geometric } O(\gamma^N), \gamma \in (0; 1)$

$\text{Convergence to an expected value}$

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$\text{Importance sampling distribution } p(x)$

$\text{Proposal distribution } T_{i \rightarrow j}$

$\text{Variance of the estimate}$

$\text{Acceptance rate, correlation of samples}$

$\text{Number of samples}$

$\text{Number of moves (mutations)}$
Metropolis Light Transport
Image Generation

- Reduce per-pixel integrals to a single integral
  - Each pixel has an individual filter function then
- Compute the distribution over the image plane
  - Bin this distribution into corresponding pixels
- Walk over the image plane
Metropolis Light Transport

- State space = space of full paths, *path space*
- What is the function $f$ for light transport?
- Interested in flux arriving at image plane
Measurement Contribution

- Measurement contribution $f$ for $k$-length path

$$f(\bar{x}_k) = L_e G \left( \prod_{k-1} \rho_k G_k \right) W_e$$
Measurement Contribution

\[ f(\bar{x}_k) = \prod_k \frac{dQ}{dA_k} = \frac{dQ}{d\mu_k} \quad [W/(m^2)^k] \]

- Flux through all differential areas of a path
Comparing Paths

- MH needs to compare two states (paths)
  - Use flux through the infinitesimal path beam

- Directly comparable for equal-length paths
  - Compare flows of energy through each path

- For different lengths the measure is different
  - Always compare fluxes going through each path
Path Integral

- For path of length \( k: I_k = \int_{\Omega_k} f(\bar{x})d\mu_k(\bar{x}) \)

- Combine all path lengths into a single integral
  - Use unified measure for all paths
    \[ d\mu(D) = \sum_{k=1}^{\infty} d\mu_k(D \cap \Omega_k) \]
  - Compare paths of different length
  - Compare groups of paths

- Use \( f \) in Metropolis-Hastings!
Metropolis Light Transport

1. Generate initial path $\bar{x}_0$ using PT/BDPT
2. Mutate with some proposal distribution $T_{\bar{x}_i \rightarrow \bar{x}_j}$
3. Accept new path $\bar{x}_j$ with probability $a_{\bar{x}_i \rightarrow \bar{x}_j}$
4. Accumulate contribution to the image plane
5. Go to step 2
Advantages

- More robust to complex light paths
  - Remembers successful paths
- Utilizes coherence of image pixels
  - Explores features faster
- Cheaper samples
  - Correlated
- Flexible path generators (mutations)
Energy redistribution path tracing [Cline05]

- Run many short Markov chains for each seed
- Adaptive number of chains according to path energy
- In spirit of Veach's lens mutation
Normalization and Start-up Bias in MLT
Differences to MCMC

- We *do* have a good alternative sampler
  - Path tracer / bidirectional path tracer
  - Easy to compute normalization constant
- No start-up bias, start within the equilibrium
  - Start many chains stratified over path space
  - Scales well with massively parallel MLT
Mutation Strategies and Their Properties
Good Mutation Criteria

- Lightweight mutation: change a few vertices
- Low correlation of samples
  - Large steps in path space
- Good stratification over the image plane
  - Hard to control, usually done by re-seeding
- It’s OK to have many specialized mutations
Existing Mutation Strategies
Veach Mutations

- Minimal changes to the path
  - Lens, caustics, multi-chain perturbations
- Large changes to the path
  - Bidirectional mutation
    - BDPT-like large step
  - Lens mutation
    - stratified seeding on the image plane
Kelemen Mutation

- Mutate a “random” vector that maps to a path
- Symmetric perturbation of “random” numbers
- Use the “random” vector for importance pdfs
  - Primary space: importance function domain
  - Assume the importance sampling is good
Kelemen Mutation, Part II

- Acceptance probability $a_{i \rightarrow j} = (f_j/p_j)/(f_i/p_i)$
  - Easy to compute: just take values from PT/BDPT

- Large step: pure PT / BDPT step
  - Generate primary sample (random vector) anew
Manifold Exploration Mutation

- Works in the local parameterization of current path
- Can connect through a specular chain
- Freezes integration dimensions
  - Tries to keep $f$ constant by obeying constraints
Combinations

- Manifold exploration can be combined
  - With Veach mutation strategies in MLT
  - With energy redistribution path tracing
- Combine Kelemen’s and Veach’s mutations?
  - Possible, yet unexplored option
Population Monte Carlo
Light Transport
Population Monte Carlo Framework

- Use a *population* of Markov chains
  - Can operate on top of Metropolis-Hastings
- Rebalance the workload
  - Weakest chains are eliminated
  - Strongest chains are forked into multiple
- Use mixture of mutations, adapt to the data
  - Select optimal mutation on the fly
Population Monte Carlo ERPT [Lai07]

- Spawn a population of chains with paths
  - Do elimination and reseeding based on path energy
- Use many mutations with different parameters
  - Reweight them on-the-fly based on the efficiency
  - Lens and caustics perturbations in the original paper
- We will show PMC with manifold exploration
Thank You for Your attention.

Part one questions?