Error Analysis of the Hemispherical SH Projection

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Abstract

Objective of this paper is to clarify the error analysis for hemispherical projections given in the Appendix of Sloan et al.'s SIGGRAPH 2003 paper "Clustered Pricncipal Components…" [1]. The first section gives some basic formulae, section 2 details the derivation of the formula for the error of the zero hemisphere projection. The last two sections are the transcription of my e-mail conversation with John Snyder and it gives some further explanations of the error analysis. This is not a stand alone paper. It will make sense only after reading the Appendix of [1].

1 Prerequisites

This section gives some basic formulae and theorems required for understanding the rest of the paper.

Square of a Sum

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i\right)^2 &= (p_1 + p_2 + \ldots + p_n) \cdot (p_1 + p_2 + \ldots + p_n) \\ &= p_1(p_1 + p_2 + \ldots + p_n) + \ldots + p_n(p_1 + p_2 + \ldots + p_n) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j. \end{aligned} \quad (1)$$

Integrating Square of a SH-Represented Function over the Hemisphere

$$\begin{aligned}\int_H \left[\sum_i c_i Y_i(s)\right]^2 \mathrm{d}s &= \int_H \sum_i \sum_j (c_i Y_i(s) Y_j(s) c_j) \mathrm{d}s \\ &= \sum_i \sum_j c_i \left(\int_H Y_i(s) Y_j(s) \mathrm{d}s\right) c_j \\ &= \sum_i \sum_j c_i A_{ij} c_j \\ &= \mathbf{c}^T \mathbf{A} \mathbf{c} \end{aligned} \quad (2)$$

Equation (1) was used in the first step. Notation:

$Y_i(s)$	i-th spherical harmonics basis function,
$\mathbf{c} = \{c_i\}$	coefficient vector,
$\mathbf{A} = [A_{ij}]$	matrix of products of SH basis functions integrated over the hemisphere.

Matrix $\mathbf{A}$ is real symmetric. Note that if the integration domain were the full sphere, matrix $\mathbf{A}$ would be the identity (SH is orthonormal) and the integral would be equal to $\mathbf{c} \cdot \mathbf{c}$.

Spectral Theorem Any real symmetric matrix $\mathbf{M}$ can be diagonalized

$$\mathbf{M} = \mathbf{R}^T\mathbf{D}\mathbf{R},$$

where $\mathbf{R}$ is a rotation matrix (i.e. real orthonormal matrix) and $\mathbf{D}$ is diagonal. For any rotation matrix $\mathbf{R}^T = \mathbf{R}^{-1}$ and thus $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

2 More Detailed Derivation of E_1

This section details the derivation of the formula for E_1 on page 8. E_1 is the $L2$ distance *over the upper hemisphere* between two different SH approximations of the same function: $\tilde{g}_c(s) = \sum_i c_i Y_i(s)$ and $\tilde{g}_b(s) = \sum_i b_i Y_i(s)$. The two coefficient vectors $\mathbf{c}$ and $\mathbf{b}$ are related by $\mathbf{b} = \mathbf{A}\mathbf{c}$, where $\mathbf{A}$ is the same matrix as in Equation (2).

$$\begin{aligned}
E_1 &= \|\tilde{g}_b - \tilde{g}_c\|_{L2} \\
&= \int_H [\tilde{g}_b(s) - \tilde{g}_c(s)]^2 \mathrm{d}s \\
&= \int_H [\sum_i b_i Y_i(s) - \sum_i c_i Y_i(s)]^2 \mathrm{d}s \\
&= \int_H [\sum_i (b_i - c_i) Y_i(s)]^2 \mathrm{d}s \\
&= (\mathbf{b} - \mathbf{c})^T \mathbf{A} (\mathbf{b} - \mathbf{c}) \qquad \text{(Equation (2))} \\
&= (\mathbf{A}\mathbf{c} - \mathbf{c})^T \mathbf{A} (\mathbf{A}\mathbf{c} - \mathbf{c}) \\
&= \mathbf{c}^T [(\mathbf{A} - \mathbf{I})^T \mathbf{A} (\mathbf{A} - \mathbf{I})] \mathbf{c} \\
&= \mathbf{c}^T \mathbf{Q}_1 \mathbf{c}
\end{aligned}$$

Matrix $\mathbf{Q}_1 = (\mathbf{A} - \mathbf{I})^T \mathbf{A} (\mathbf{A} - \mathbf{I})$ is real symmetric.

3 E-mail 1: Me → John Snyder

…I have problems understanding the error error analysis for different hemispherical projections that you give in the Appendix of your SIGGRAPH 2003 paper [1]. You say

$$E_1 = \ldots = \mathbf{c}^T \mathbf{Q}_1 \mathbf{c}$$

and you are looking for the maximum of E_1 over all $||\mathbf{c}|| = 1$. You say that this is equal to the maximum singular value of the matrix $\mathbf{Q}_1$.

This is very similar to the standard definition of the matrix 2-norm given in linear algebra textbooks. The 2-norm of a matrix $\mathbf{Q}$ is defined as the maximum of $\|\mathbf{Qc}\|$ over all $||\mathbf{c}|| = 1$, i.e.

$$\|\mathbf{Q}\| = \max_{\|\mathbf{c}\|=1} \|\mathbf{Qc}\|.$$

Textbooks also say that this norm is equal to the maximum singular value of $\mathbf{Q}$. But I do not know how to relate

$$\underbrace{\max_{\|\mathbf{c}\|=1} \mathbf{c}^T\mathbf{Qc}}_{\text{(from your paper)}} \quad = \quad \underbrace{\max_{\|\mathbf{c}\|=1} \|\mathbf{Qc}\|}_{\text{(from linear algebra books)}} .$$

For the average error, how comes that it is equal to the average of the singular values? Could you give me a reference please?

I also did not understand why, for the odd projection, you use only the upper-left block of $\mathbf{Q}_2$ for the average error analysis.

Could you please clarify those points to me.

4 Reply from John Snyder

It's simple to show that the two norms are identical for a *symmetric matrix*. Using the SVD[1], we have

$$\mathbf{Q} = \mathbf{R}^T\mathbf{DR}$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{D}$ is a diagonal matrix with the singular values on the diagonal. Then

$$\mathbf{c}^T\mathbf{Qc} = \mathbf{c}^T\mathbf{R}^T\mathbf{DRc} = \mathbf{c}'^T\mathbf{Dc}'$$

where since $\mathbf{R}$ is a rotation matrix, $\mathbf{c}'$ is also unit length. Thus the max value attained is given by the largest component of $\mathbf{D}$ (largest singular value), for the vector $\mathbf{c}$ given by the first row of $\mathbf{R}$, if $\mathbf{D}$ is arranged with its first component as the largest singular value.

Similarly,

$$\begin{aligned}
\|\mathbf{Qc}\| &= [(\mathbf{Qc})^T(\mathbf{Qc})]^{1/2} \\
&= [\mathbf{c}^T\mathbf{Q}^T\mathbf{Qc}]^{1/2} \\
&= [\mathbf{c}^T\mathbf{R}^T\mathbf{DRR}^T\mathbf{DRc}]^{1/2} \\
&= [\mathbf{c}^T\mathbf{R}^T\mathbf{D}^2\mathbf{Rc}]^{1/2} \\
&= [\mathbf{c}'^T\mathbf{D}^2\mathbf{c}]^{1/2}
\end{aligned}$$

But $\mathbf{D}^2$ is just the diagonal matrix with the squares of the singular values on the diagonal. So again the max value achieved for $\mathbf{c}'^T\mathbf{D}^2\mathbf{c}'$ is the square of the max singular value. Then after the square root, we get the max singular value back again.

To show that the average error is the average singular value, I did an analytic integration using the canonical parameterization of an n-dimensional sphere. I suspect this is a standard result in

[1] Singular Value Decomposition, see also the Spectral Theorem in Section 1

linear algebra, but I looked through a couple books today and didn't manage to find it. I can go through this in more detail if necessary.

As for $\mathbf{Q}_2$, I take the upper left 25×25 block because the optimal input vector $\mathbf{c}$ is still 25D. I can see how you are confused since I skipped a couple steps in the derivation due to space limitations.

Given a $\mathbf{c}$ (25D, unit length, optimal hemispherical projection), the odd hemispherical projection is given by

$$\mathbf{b}_{odd} = \mathbf{D}^* \mathbf{A} \mathbf{c}$$

Think of $\mathbf{c}$ here as being padded with 0's at the bottom so what was initially a 25D vector becomes a 64D one. Then the error is given by

$$E_2 = \mathbf{c}^T \mathbf{Q}_2 \mathbf{c}$$

where again, the c's are padded with 0's from coef 26 to 64. (Note that $\mathbf{b}_{odd}$ will not be so padded!) So we only need the upper left 25×25 of $\mathbf{Q}_2$, though we clearly need the full 64×64 matrices $\mathbf{A}$ and $\mathbf{D}^*$ to derive it and also to derive the actual odd hemispherical projection vector $\mathbf{b}_{odd}$.

To show why $\mathbf{b}_{odd} = \mathbf{D}^* \mathbf{A} \mathbf{c}$, note that $\mathbf{A}\mathbf{c}$ is the zero-hemispherical projection (in this case, a 64D one). Here I'm just rewriting $\mathbf{c} = \mathbf{A}^{-1}\mathbf{b}$, which derives the optimal vector in terms of the zero-hemisphere one. Clearly $\mathbf{D}^*$ must zero out the even basis functions, since we're only using the odd ones here. Since we don't care what happens on the $z < 0$ hemisphere, we double the odd basis functions, which reproduces the zero-hemisphere projection onto the part of the sphere we care about, and does a negative reflection to the part of the sphere we don't care about and don't measure in the error analysis.

References

[1] Peter-Pike Sloan, Jesse Hall, John Hart, and John Snyder. Clustered principal components for precomputed radiance transfer. *ACM Trans. Graph.*, 22(3):382–391, 2003.