Radiant Flux Emitted by a VRML SpotLight-like Luminaire

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November 27, 2006

Consider a point light source whose radiant intensity is given by the following equation:

$$I(\theta,\phi) = \begin{cases} I_0 & \theta \leq \alpha \\ I_0 \dfrac{\beta-\theta}{\beta-\alpha} & \alpha < \theta < \beta \\ 0 & \theta \geq \beta \end{cases}$$

where α and β are the light source parameters ($\beta \geq \alpha$). In other words, for θ between zero and α, the intensity is constant I_0. Then it linearly falls off to zero at $\theta = \beta$. Such a light source is equivalent to the VRML SpotLight with $\mathsf{direction} = (0, 0, 1)$, $\mathsf{beamWidth} = \alpha$, and $\mathsf{cutOffAngle} = \beta$.

We want to compute the total radiant flux emitted by this light source.

Constant part

$$\Phi_1 = \int_0^{2\pi} \int_0^{\alpha} I_0 \sin\theta d\theta d\phi = I_0 2\pi(1 - \cos\alpha).$$

Linear part

$$\Phi_2 = \int_0^{2\pi} \int_\alpha^{\beta} I_0 \frac{\beta-\theta}{\beta-\alpha} \sin\theta d\theta d\phi = I_0 \frac{2\pi}{\beta-\alpha} \int_\alpha^{\beta} (\beta-\theta)\sin\theta d\theta \tag{1}$$

The last integral is the sum of the following two integrals:

$$\int_\alpha^{\beta} \beta \sin\theta d\theta = \beta\cos\alpha - \beta\cos\beta \tag{2}$$

$$-\int_\alpha^{\beta} \theta\sin\theta d\theta = \Big| \sin\theta - \theta\cos\theta \Big|_\beta^\alpha = \sin\alpha - \alpha\cos\alpha - \sin\beta + \beta\cos\beta \tag{3}$$

Plugging (2) and (3) into (1) and rearranging, we get

$$\Phi_2 = I_0 \frac{2\pi}{\beta-\alpha} \left[(\beta-\alpha)\cos\alpha + \sin\alpha - \sin\beta\right] = I_0 2\pi \left[\cos\alpha - \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\right]. \tag{4}$$

Total flux

$$\boxed{\Phi = \Phi_1 + \Phi_2 = I_0 2\pi \left[1 - \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\right]} \tag{5}$$

Isn't the final formula beautiful?