Radiant Flux Emitted by a VRML SpotLight-like Luminaire

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Consider a point light source whose radiant intensity is given by the following equation:

\[
I(\theta, \phi) = \begin{cases} 
I_0 & \theta \leq \alpha \\
I_0 \frac{\beta - \theta}{\beta - \alpha} & \alpha < \theta < \beta \\
0 & \theta \geq \beta 
\end{cases}
\]

where \(\alpha\) and \(\beta\) are the light source parameters \((\beta \geq \alpha)\). In other words, for \(\theta\) between zero and \(\alpha\), the intensity is constant \(I_0\). Then it linearly falls off to zero at \(\theta = \beta\). Such a light source is equivalent to the VRML SpotLight with direction \(= (0, 0, 1)\), beamWidth \(= \alpha\), and cutOffAngle \(= \beta\).

We want to compute the total radiant flux emitted by this light source.

**Constant part**

\[
\Phi_1 = \int_0^{2\pi} \int_0^\alpha I_0 \sin \theta d\theta d\phi = I_0 2\pi (1 - \cos \alpha).
\]

**Linear part**

\[
\Phi_2 = \int_0^{2\pi} \int_\alpha^\beta I_0 \frac{\beta - \theta}{\beta - \alpha} \sin \theta d\theta d\phi = I_0 \frac{2\pi}{\beta - \alpha} \int_\alpha^\beta (\beta - \theta) \sin \theta d\theta
\]

The last integral is the sum of the following two integrals:

\[
\int_\alpha^\beta \beta \sin \theta d\theta = \beta \cos \alpha - \beta \cos \beta
\]

\[
-\int_\alpha^\beta \theta \sin \theta d\theta = \left| \sin \theta - \cos \theta \right|_\alpha^\beta = \sin \alpha - \alpha \cos \alpha - \sin \beta + \beta \cos \beta
\]

Plugging (2) and (3) into (1) and rearranging, we get

\[
\Phi_2 = I_0 \frac{2\pi}{\beta - \alpha} [((\beta - \alpha) \cos \alpha + \sin \alpha - \sin \beta)] = I_0 2\pi \left[ \cos \alpha - \frac{\sin \beta - \sin \alpha}{\beta - \alpha} \right].
\]

**Total flux**

\[
\Phi = \Phi_1 + \Phi_2 = I_0 2\pi \left[ 1 - \frac{\sin \beta - \sin \alpha}{\beta - \alpha} \right]
\]

Isn’t the final formula beautiful?