

# Radiant Flux Emitted by a VRML SpotLight-like Luminaire

Jaroslav Křivánek

November 27, 2006

Consider a point light source whose radiant intensity is given by the following equation:

$$I(\theta, \phi) = \begin{cases} I_0 & \theta \leq \alpha \\ I_0 \frac{\beta - \theta}{\beta - \alpha} & \alpha < \theta < \beta \\ 0 & \theta \geq \beta \end{cases}$$

where  $\alpha$  and  $\beta$  are the light source parameters ( $\beta \geq \alpha$ ). In other words, for  $\theta$  between zero and  $\alpha$ , the intensity is constant  $I_0$ . Then it linearly falls off to zero at  $\theta = \beta$ . Such a light source is equivalent to the VRML SpotLight with direction = (0, 0, 1), beamWidth =  $\alpha$ , and cutOffAngle =  $\beta$ .

We want to compute the total radiant flux emitted by this light source.

## Constant part

$$\Phi_1 = \int_0^{2\pi} \int_0^\alpha I_0 \sin \theta d\theta d\phi = I_0 2\pi (1 - \cos \alpha).$$

## Linear part

$$\Phi_2 = \int_0^{2\pi} \int_\alpha^\beta I_0 \frac{\beta - \theta}{\beta - \alpha} \sin \theta d\theta d\phi = I_0 \frac{2\pi}{\beta - \alpha} \int_\alpha^\beta (\beta - \theta) \sin \theta d\theta \quad (1)$$

The last integral is the sum of the following two integrals:

$$\int_\alpha^\beta \beta \sin \theta d\theta = \beta \cos \alpha - \beta \cos \beta \quad (2)$$

$$- \int_\alpha^\beta \theta \sin \theta d\theta = \left| \sin \theta - \theta \cos \theta \right|_\beta^\alpha = \sin \alpha - \alpha \cos \alpha - \sin \beta + \beta \cos \beta \quad (3)$$

Plugging (2) and (3) into (1) and rearranging, we get

$$\Phi_2 = I_0 \frac{2\pi}{\beta - \alpha} [(\beta - \alpha) \cos \alpha + \sin \alpha - \sin \beta] = I_0 2\pi \left[ \cos \alpha - \frac{\sin \beta - \sin \alpha}{\beta - \alpha} \right]. \quad (4)$$

## Total flux

$$\Phi = \Phi_1 + \Phi_2 = I_0 2\pi \left[ 1 - \frac{\sin \beta - \sin \alpha}{\beta - \alpha} \right] \quad (5)$$

Isn't the final formula beautiful?