Realtime Computer Graphics on GPUs

Curves and Surfaces

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Overview

Curve and surface definition

Explicit : $y = sin(x),\ z = x^2 + y^2$

Implicit : $0 = x^2 + y,\ 0 = x^2 + y^2 + z^2$

Parametric :

- $\mathbf{Q}(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$
- $\mathbf{P}(u,v) = \begin{bmatrix} cos(u)(R + r * cos(v)) \\ sin(u)(R + r * cos(v)) \\ r * sin(v) \end{bmatrix}$

PARAMETRIC CURVES AND SURFACES

- Parameter(s) have specified range ($[0, 1]$)
- Same curve (surface)can have multiple parametrizations
  - Arc length parametrization - animation, uniform sampling, ...
  - In general case cannot be expressed analyticaly
- Sampling the parameter space $\rightarrow$ discrete points in space
  - Approximating by polyline or polygonal mesh

Cubic Curves

Cubic Curves – Why?

- Simple to work with
- Physical motivation (natural spline)
- Humans do not need higher order derivatives to perceive smoothness

Cubic Curves

Basic parametric cubic curve equation:

$$\mathbf{Q}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2 + \mathbf{d}t^3$$

CUBIC CURVES

Basic parametric cubic curve equation:

$$
\begin{aligned}
Q_x(t) &= a_x + b_x t + c_x t^2 + d_x t^3 \\
Q_y(t) &= a_y + b_y t + c_y t^2 + d_y t^3 \\
Q_z(t) &= a_z + b_z t + c_z t^2 + d_z t^3
\end{aligned}
$$

Cubic Curves

Rewrite in a matrix form:

$$\mathbf{Q}(t) = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

CUBIC CURVES

Rewrite in a matrix form:

$$\mathbf{Q}(t) = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Compact matrix form:

$$\mathbf{Q}(t) = \mathbf{C}\mathbf{T}(t)$$

C is coefficient matrix (constant)

CUBIC CURVES

Simple derivative:

$$\mathbf{Q}'(t) = \mathbf{C}\frac{d}{dt}\mathbf{T}(t) = \mathbf{C}\begin{bmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

GEOMETRICAL CONSTRAINTS

- Certain curves can be defined as weighted sum of four geometrical constraints.
- B_k – blending functions

$$
\begin{aligned}
\mathbf{Q}(t) &= \sum_{k=0}^{3} B_k(t)\mathbf{g_k} \\
&= (a_1 + b_1 t + c_1 t^2 + d_1 t^3)\mathbf{g_1} \\
&+ (a_2 + b_2 t + c_2 t^2 + d_2 t^3)\mathbf{g_2} \\
&+ (a_3 + b_3 t + c_3 t^2 + d_3 t^3)\mathbf{g_3} \\
&+ (a_4 + b_4 t + c_4 t^2 + d_4 t^3)\mathbf{g_4}
\end{aligned}
$$

GEOMETRICAL CONSTRAINTS

Rewrite in a matrix form:

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{g_1} & \mathbf{g_2} & \mathbf{g_3} & \mathbf{g_4} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

GEOMETRICAL CONSTRAINTS

Rewrite in a matrix form:

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{g_1} & \mathbf{g_2} & \mathbf{g_3} & \mathbf{g_4} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Compact matrix form:

$$\mathbf{Q}(t) = \mathbf{GMT}(t)$$

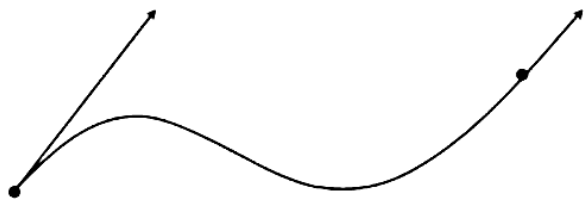
$\mathbf{G}$ geometry matrix, $\mathbf{M}$ basis matrix

Hermite Curves

- Endpoints $\mathbf{P_1}, \mathbf{P_2}$
- Tangents at those points $\mathbf{T_1}, \mathbf{T_2}$

$$\mathbf{G} = \begin{bmatrix} \mathbf{P_1} & \mathbf{P_2} & \mathbf{T_1} & \mathbf{T_2} \end{bmatrix}$$

$$\mathbf{M_H} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

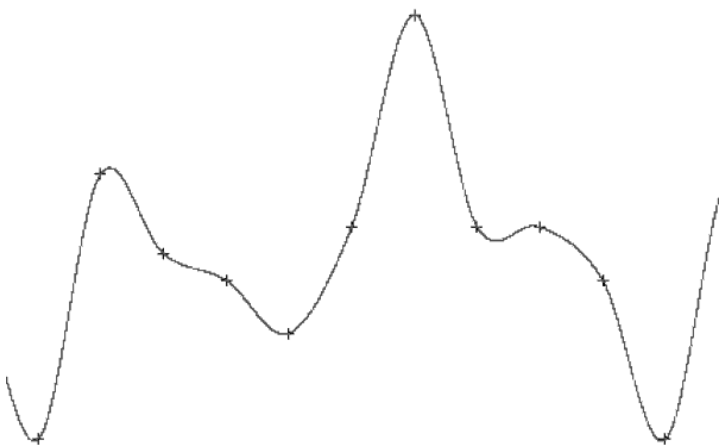


CATMULL-ROM SPLINES

- Control points $\mathbf{P_1}, \ldots \mathbf{P_n}$
- Tangent at point: $\mathbf{T_i} = \frac{1}{2}(\mathbf{P_{i+1}} - \mathbf{P_{i-1}}$
- Catmull-Rom $\rightarrow$ Hermite

$$\mathbf{G_H} = \begin{bmatrix} \mathbf{P_{i-1}} & \mathbf{P_i} & \mathbf{P_{i+1}} & \mathbf{P_{i+2}} \end{bmatrix}$$

$$\mathbf{M_{CR}} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 2 & -1 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



Bézier Curves

- P. de Casteljau (Citroen) – numerically stable algorithm using linear interpolations
- P. Bézier (Renault) – Bernstein polynomials as blending functions

$$B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

- Four control points per segment

$$\mathbf{B}(t) = \sum_{k=0}^{3} B_{3,k}(t)\mathbf{P}_k$$

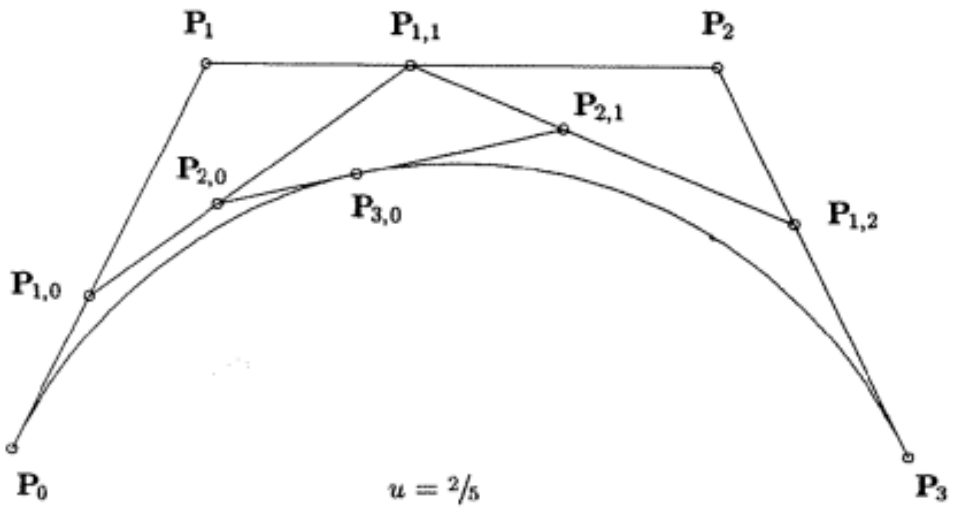
Basis matrix

$$\mathbf{G_B} = \begin{bmatrix} \mathbf{P_1} & \mathbf{P_2} & \mathbf{P_3} & \mathbf{P_4} \end{bmatrix}$$

$$\mathbf{M_B} = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DE CASTELJAU ALGORITHM

- Repeated linear interpolation
- Each *lerp* uses same weight



Surfaces

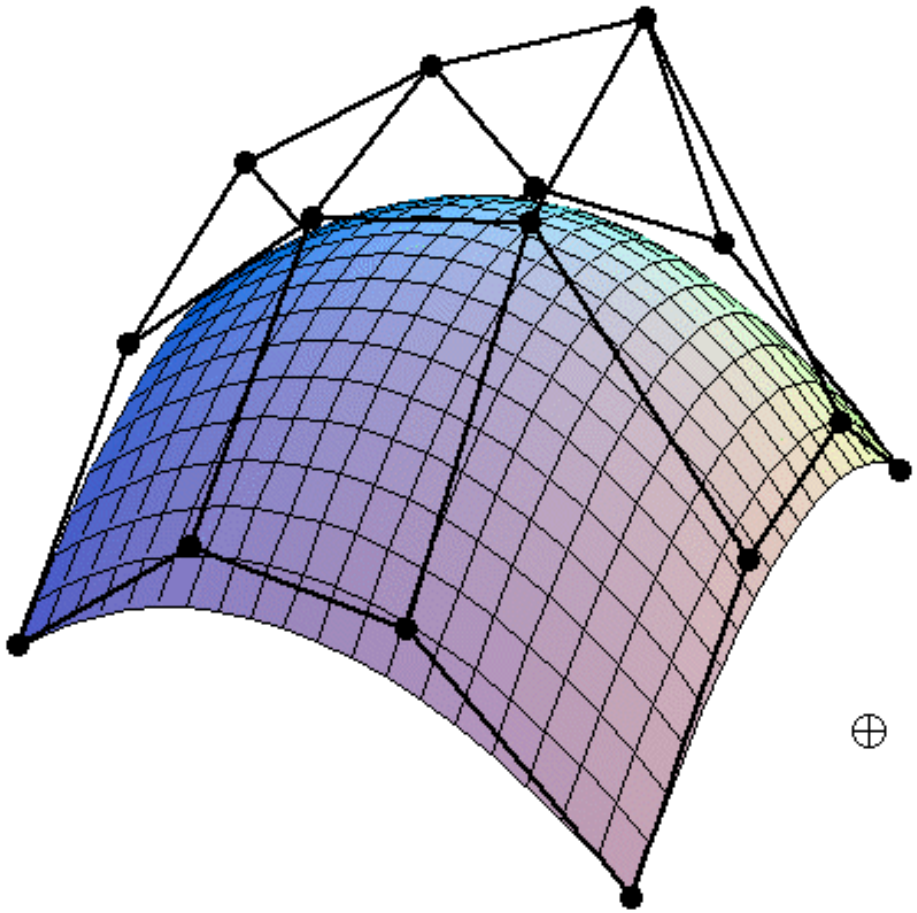
Bicubic Surfaces

- Extension of bilinear interpolation concept
- Tensor product surfaces
- 16 geometrical constraints - $\mathbf{G} - 4 \times 4 \times 3$ coefficients

$$\mathbf{Q}(u,v) = \sum_{k=0}^{3} \sum_{l=0}^{3} B_k(u) B_l(v) \mathbf{g}_{kl}$$

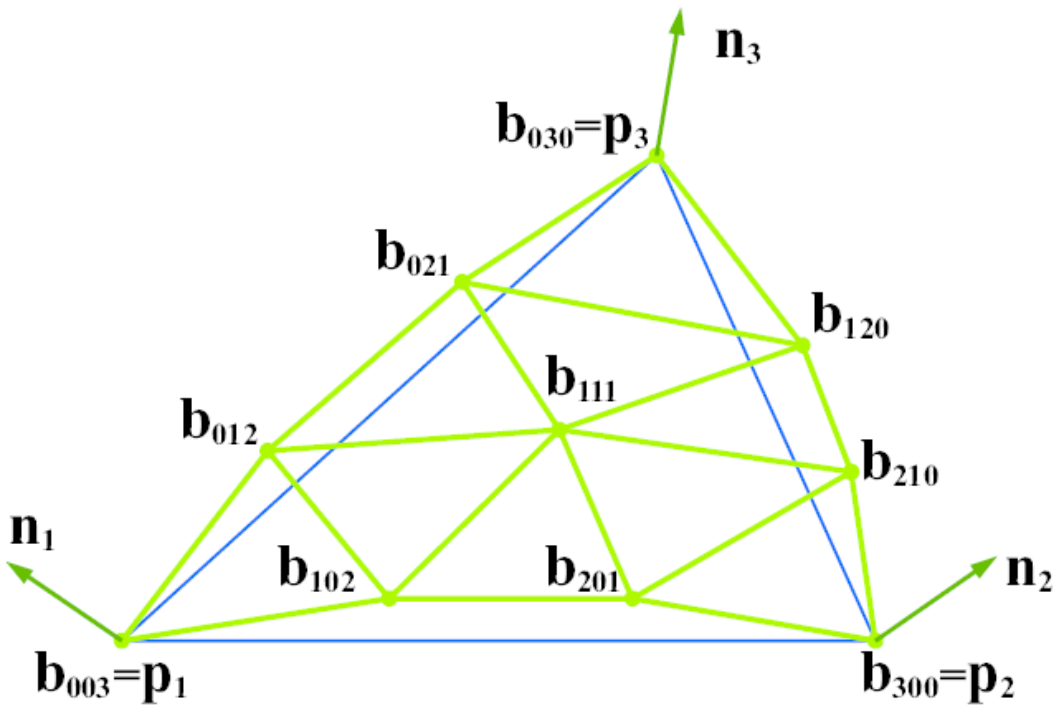
$$Q^r(u,v) = \mathbf{S}^T(v) \mathbf{M}^t \mathbf{G}^r \mathbf{M} \mathbf{T}(u)$$

Bézier Patch



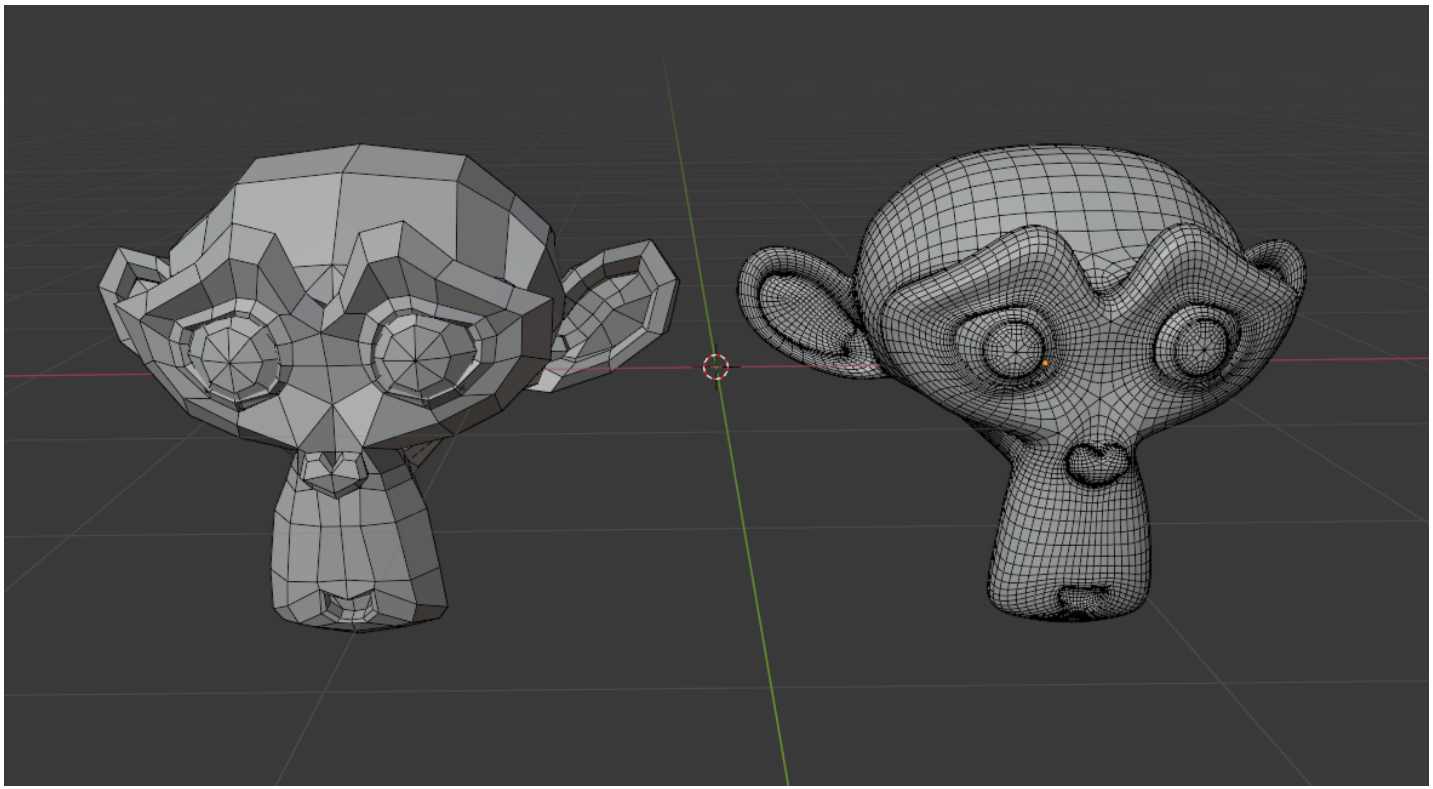
Bézier Triangle

- de Casteljau extension for triangles - barycentric coordinates instead of lerp
- 10 control points
  - Simplification for triangle meshes with vertex normals – PN-triangles



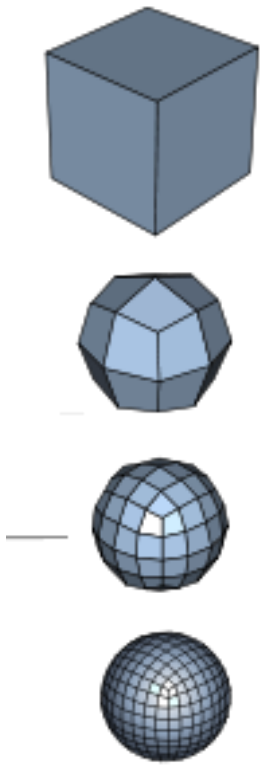
Subdivision Surfaces

- Recursive algorithms
- Two steps:
  - Refine mesh topology
  - Adjust vertex positions
- Interpolating vs approximating



CATMULL-CLARK SCHEME

- Produces quad mesh - keeps clean topology
- Vertices inserted into edges and face centers



Loop Scheme

- Defined for triangle meshes
- Spliting edges
- New position - weighted average of vertices from incident triangles