Quaternions
&
Computer Animation

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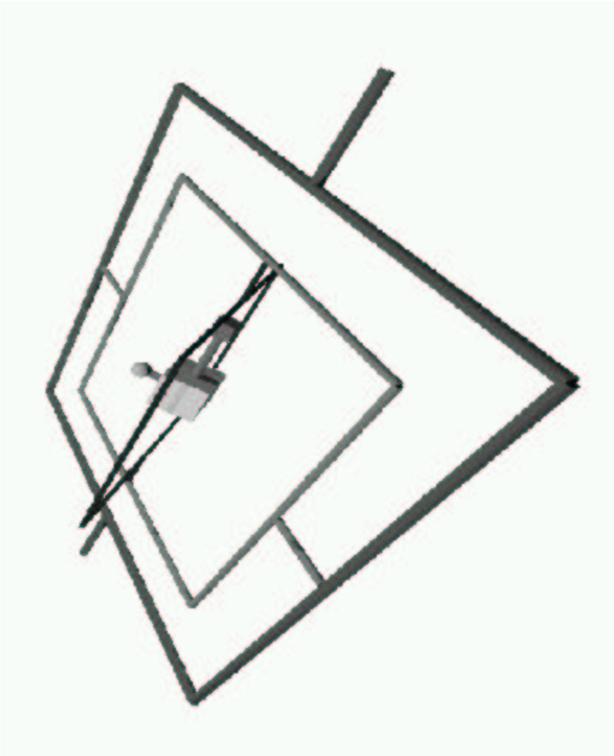
Quaternions – Historical Review

- Generalized complex numbers
- William Rowan Hamilton, 1843
- Rotation in 3D space
- 3D numbers failed
- Scalar and vector part

Euler Angles

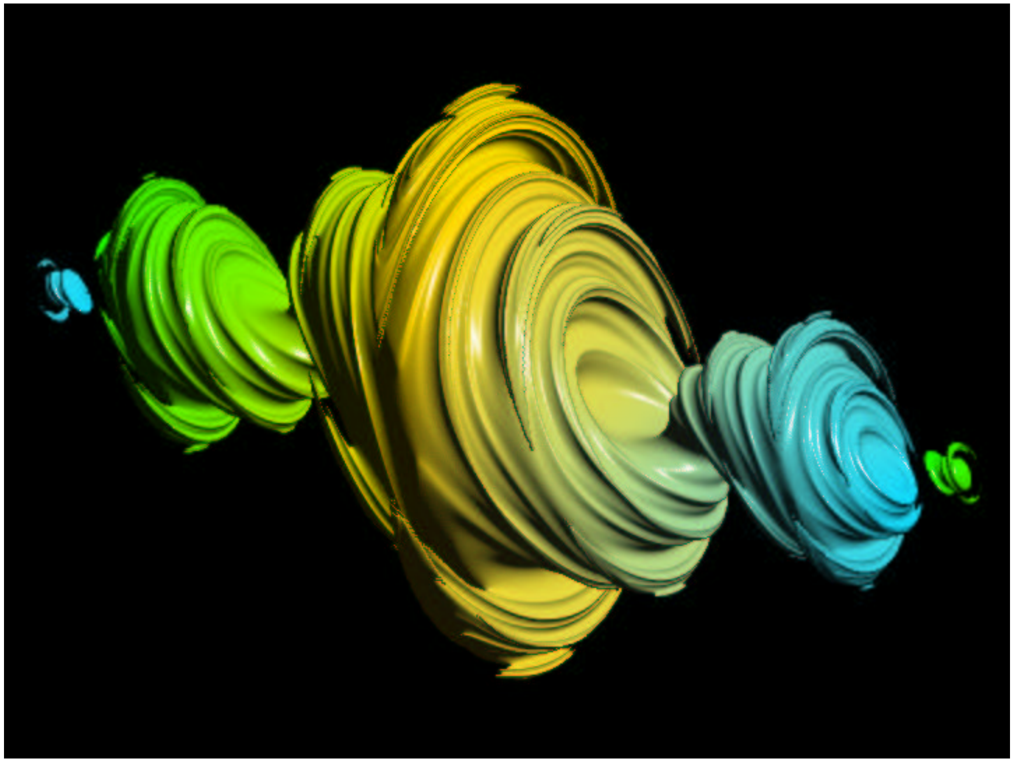
- Rotation about axes – 3 angles
- Order–dependent
- Lack of intuition
- Gimbal Lock
- Complicated interpolation

(Homogenous) Rotation Matrix



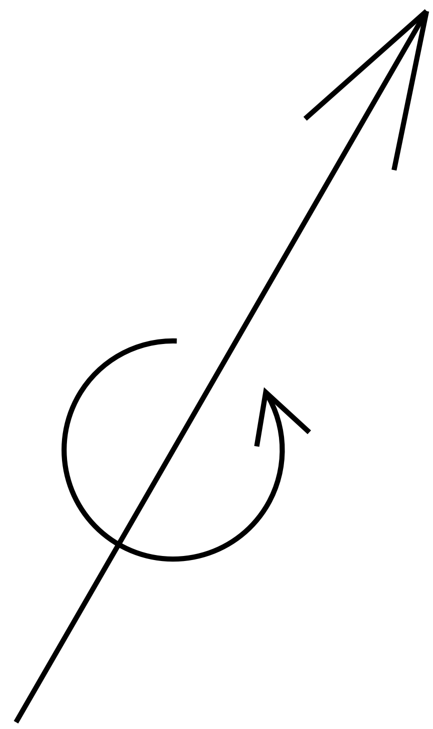
Applications

- 3D Rotation
- 3D (figure) Animation
- 3D/4D fractals
- Wavelets
- . . .



Quaternion

- $q = \langle s, x, y, z \rangle = \langle s, v \rangle \in \mathbb{H}$
- $q = s + ix + jy + kz$
- $i^2 = j^2 = k^2 = ijk = -1$
  $ij = k, \qquad ji = -k$
- Scalar $a \in \mathbb{R} \Rightarrow \langle a, 0 \rangle \in \mathbb{H}$
  Vector $v \in \mathbb{R}^3 \Rightarrow \langle 0, v \rangle \in \mathbb{H}$



Quaternion Mathematics

- Addition $(s + ix + jy + kz) + (t + ia + jb + kc)$
- Multiplication $\langle s, v \rangle \langle t, u \rangle = \langle st - vu, v \times u + su + tv \rangle$
  $$\langle s, 0 \rangle \langle t, u \rangle = \langle st, su \rangle = \langle t, u \rangle \langle s, 0 \rangle$$
- Conjugation $\langle s, v \rangle^* = \langle s, -v \rangle$
- Inner product $\langle s, v \rangle \cdot \langle t, u \rangle = \langle st, v \cdot u \rangle$
- Size $||q|| = \sqrt{q \cdot q} = \sqrt{qq^*}$
- Division $q^{-1} = \frac{q^*}{||q||^2} \qquad \frac{p}{q} = pq^{-1}$

Unit Quaternions

- 4D Sphere $\mathbb{H}_1 = \{q \in \mathbb{H}; \quad ||q|| = 1\}$
- $p, q \in \mathbb{H}_1 \Rightarrow ||pq|| = 1, \quad p^{-1} = p^*$
- $q \in \mathbb{H}_1 \Rightarrow \exists v \in \mathbb{R}^3, \theta \in \mathbb{R} : q = \langle \cos\theta, v \sin\theta \rangle$
- $p, q \in \mathbb{R}^4 \approx \mathbb{H} : \quad p \cdot q = ||p|| ||q|| \cos\alpha$
- Rotation of r about q: qrq^{-1}
  $q = \langle \cos\theta, v \sin\theta \rangle$: v is rotation axis and 2θ angle

Linear Interpolation Of

$$Lerp(p, q, t) = p(1 - t) + qt$$

- Euler angles, Rotation matrix – not optimal trajectory
- Quaternion – floating angular velocity
- Spherical linear quaternion interpolation – Slerp
  $Slerp(p, q, t) = p(p^*q)^t = (qp^*)^t p$

  Constant angular velocity

More keyframes

- Slerp is not smooth
- Heuristic $\times$ Mathematical approach
- Spherical spline quaternion interpolation – Squad
- $Squad(q_i, q_{i+1}, t) = Slerp(Slerp(q_i, q_{i+1}, t), Slerp(s_i, s_{i+1}, t), 2t(1-t))$
  $s_i = q_i \exp(-\frac{1}{4}(\log(q_i^{-1} q_{i+1}) + \log(q_i^{-1} q_{i-1})))$
- Spherical interpolation using numerical gradient descent – Spring

References

- Dam, Koch, Lillholm: *Quaternions, Interpolation and Animation*, Technical report, University of Copenhagen, 1998

- Shoemake: *Animating rotation with quaternion curves*, Computer Graphics, 1985

- Library *http://math3d.sourceforge.net*

- Fractals *http://www.physcip.uni-stuttgart.de/phy11733/quat_e.html*