matfyz

Computer Graphics Charles University

Mathematics for real-time graphics

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Content

- homogeneous coordinates, matrix transformations
  - coordinate-system conversions
- coordinate systems, projections, frustum
- perspective correct interpolation
- orientations
  - Euler angles, quaternions
  - orientation interpolation
- smooth interpolations and approximations
  - spline functions, natural spline, B-spline, Catmull-Rom, ...

Geometric transformations in 3D

- Cartesian 3D coordinate vector [x, y, z]
- multiplying by a 3×3 matrix
  - row vector multiplied from the right (DirectX)

$$[x, y, z] \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [x', y', z']$$

  - column vector multiplied from the left (OpenGL)
- transform matrices 3×3 have serious drawback – **cannot do translations!**

Homogeneous coordinates

- **homogeneous coordinate vector** [x, y, z, w]
- transformation: multiplying by a 4×4 matrix

$$[x, y, z, w] \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = [x', y', z', w']$$

- homogeneous matrix is able to **translate** and to do **perspective projections**

Coordinate conversions

- from **homogeneous coordinates** [x, y, z, w] into Cartesian coordinates: by division (w≠0) **[x/w, y/w, z/w]**
- coordinate vector **[x, y, z, 0]** does not correspond to any real point in space
  - can be interpreted as a **directional vector** (point in in-finity)
- from **Cartesian coordinates** to homogeneous: triv-ial extension **[x, y, z] … [x, y, z, 1]**

Elementary transformations

Affine transformation:

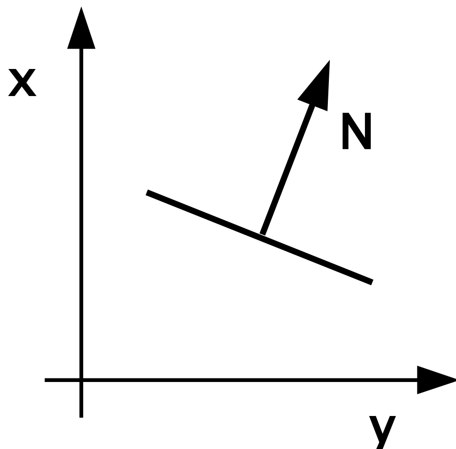
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ t_1 & t_2 & t_3 & 1 \end{bmatrix}$$

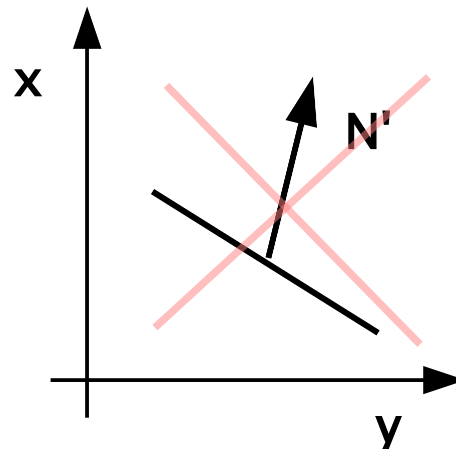
- upper left submatrix **[$\mathbf{a_{11}}$ to $\mathbf{a_{33}}$]** defines scaling, orientation and shear
- vector **[$\mathbf{t_1, t_2, t_3}$]** defines translation
  - translation is performed after former transformations

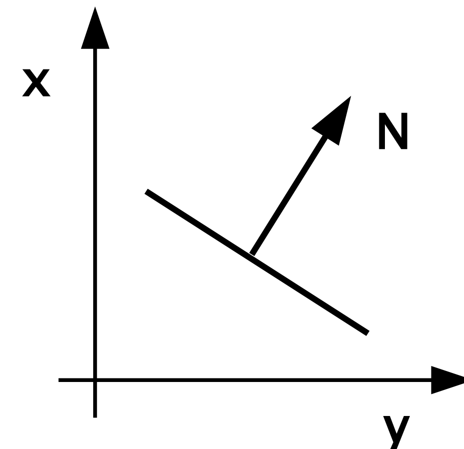
Normal vector transformation

- normal vectors must not be transformed by regular matrices (like point positions are)
  - exception: M is rotational (orthonormal)
- **normal-vector** transformation matrix **N**:

$$N = (M^{-1})^T$$







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Coordinate systems in OpenGL

Object space (object modeling) → Modeling transform (scene composition) → World space (simulation) → View transform (camera position & view) → Eye space (camera space) → Projection transform (perspective|orthographic) → Clip space (frustum = homo cube)

[x,y,z,w]

Coordinate systems in OpenGL

Clip space
(frustum = homo cube)

[x,y,z,w]

→ Perspective divide
(hardwired into HW)

→ Normalized Device Space
(frustum = cartesian cube)

[x,y,z]

→ Viewport transform
(3D scaling & translation)

→ Window space
(actual pixel coords)

[x,y,z]

−z

xy

OpenGL: [−1,−1,−1] to [1,1,1]
DirectX: [−1,−1, 0] to [1,1,1]

[x,y] actual screen coordinates (fragments)
z depth value compatible with actual depth-buffer

Coordinate systems in OpenGL

- **object space**
  - modeling of individual objects, modularity
  - 3D modeling software (3DSMax, Blender, Rhino, ..)
- **world space**
  - absolute (real) coordinates in simulated virtual world
  - object instantiation, collision detection, AI planning, ..
- **camera space**
  - the whole virtual world transforms into coordinates relative to a camera
  - center of projection: **origin**, view direction: **-z** (or **z**)

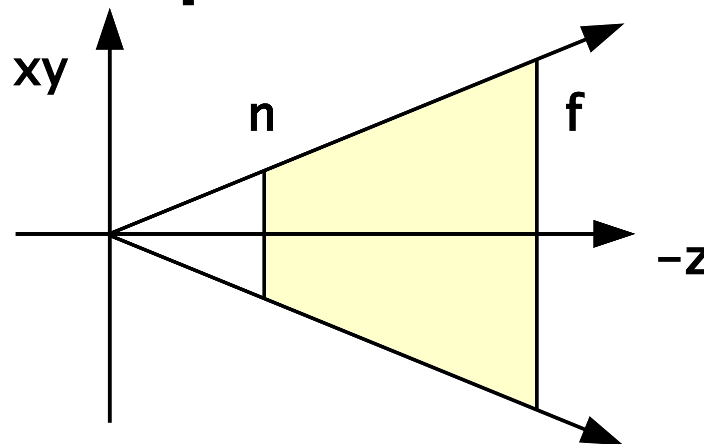
Coordinate systems & transformations

- **transformation “model → camera”**
  - altogether – “modelview” matrix
  - world coordinates are not directly used in render. pipel.
- **projection transformation**
  - defines visible volume = **frustum** [**l**, **r**, **b**, **t**, **n**, **f**]
  - front & back clip distances: **n**, **f**
  - result: homogeneous coordinate (before clipping)
- **“clip space”**
  - **mandatory output coordinate** of vertex shader !

Projection transformation

- far point **f** can be in <u>infinity</u>

$$\begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2n}{t-b} & 0 & 0 \\ -\frac{r+l}{r-l} & -\frac{t+b}{t-b} & \frac{f+n}{f-n} & 1 \\ 0 & 0 & -\frac{2fn}{f-n} & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{2n}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2n}{t-b} & 0 & 0 \\ -\frac{r+l}{r-l} & -\frac{t+b}{t-b} & 1 & 1 \\ 0 & 0 & -2n & 0 \end{bmatrix}$$

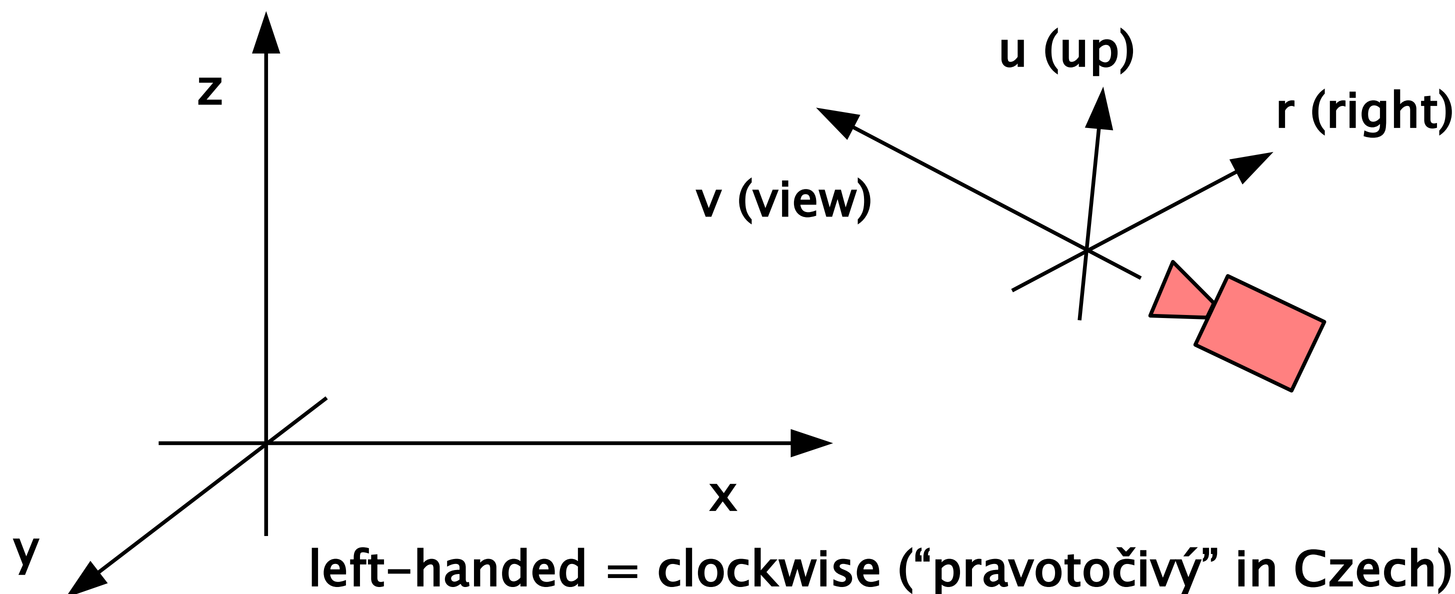


Coordinate systems & transformations

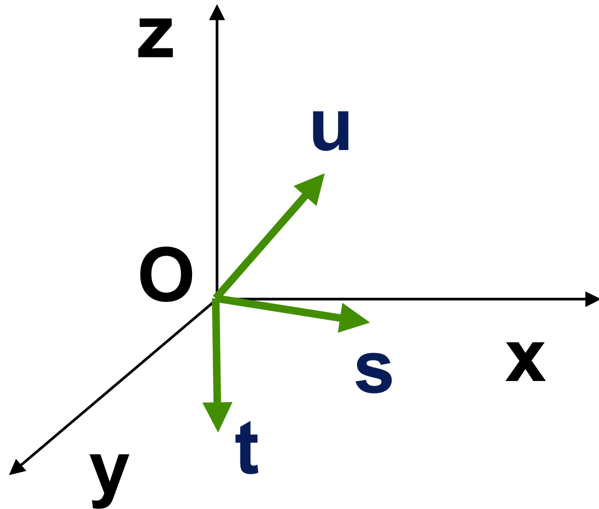
- **perspective division**
  - just converts homogeneous coordinates into Cartesian
- **normalized coordinates** ("NDS")
  - standard-sized cube/cuboid
  - OpenGL: [-1,-1,-1] to [1,1,1]
  - DirectX: [-1,-1,0] to [1,1,1]
- **window coordinates** ("window space")
  - result of **linear adjustment** to window size in pixels
  - used in **rasterizer** and all **fragment processing**

Rigid body transformation

- preserves **shapes**, alters **orientation & position**
  - translation and rotation
  - **conversion between coordinate systems** (e.g. between world-space and camera-space)



Conversion between two orientations



Coordinate system has an origin **O** and is defined by three unit vectors **[s, t, u]**

$$M_{stu \to xyz} = \begin{bmatrix} s_x & s_y & s_z \\ t_x & t_y & t_z \\ u_x & u_y & u_z \end{bmatrix}$$

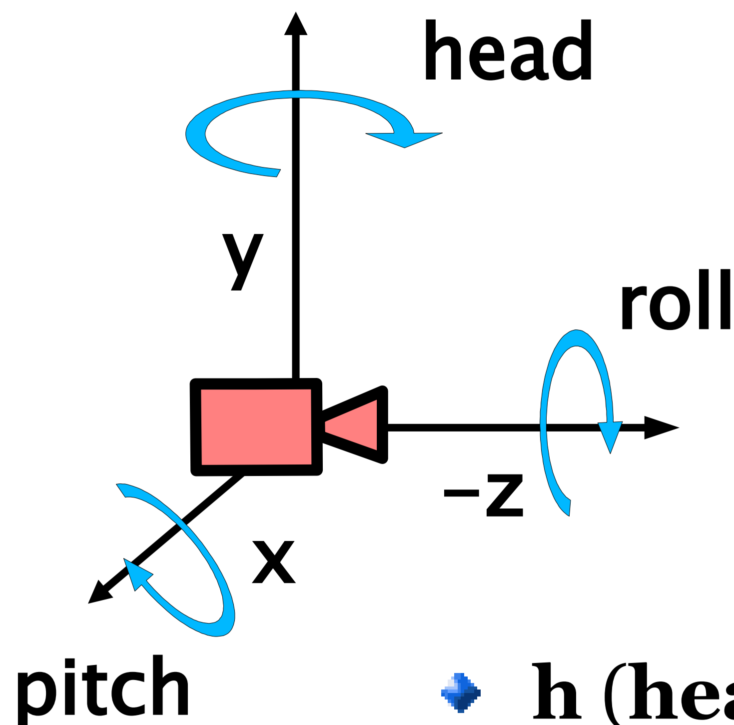
$$[1, 0, 0] \cdot M_{stu \to xyz} = s$$

$$[0, 1, 0] \cdot M_{stu \to xyz} = t$$

$$[0, 0, 1] \cdot M_{stu \to xyz} = u$$

$$M_{xyz \to stu} = M^T_{stu \to xyz}$$

Euler transformation



- arbitrary rotation decomposed into **three components**
  - Leonard Euler (1707-1783)

$$E(h,p,r) = R_y(h) \cdot R_x(p) \cdot R_z(r)$$

- **h** (**head**, yaw): plan view direction
- **p** (**pitch**): forward/backward pitching
- **r** (**roll**): rolling around the view vector

Euler transformation II

- result matrix of rotation

$$E=\begin{pmatrix} c(r)c(h)-s(r)s(p)s(h) & s(r)c(h)+c(r)s(p)s(h) & -c(p)s(h) \\ -s(r)c(p) & c(r)c(p) & s(p) \\ c(r)s(h)+s(r)s(p)c(h) & s(r)s(h)-c(r)s(p)c(h) & c(p)c(h) \end{pmatrix}$$

- s(x) .. sin(x), c(x) .. cos(x)
- backward matrix → angles computation h, p, r:
  - p .. e_{23}
  - r .. e_{21}/e_{22}
  - h .. e_{13}/e_{33}

Rotations: different conventions

- **main convention**:
  - 1. rotation around **z** by φ
  - 2. rotation around **x'** by θ
  - 3. rotation around **z''** by ψ
- **x-convention**:
  - 1. rotation around **z**
  - 2. rotation around original **x**
  - 3. rotation around original **z**
- **more systems**: aeronautics, gyroscopes, physics, ..

Quaternions

- Sir William Rowan **Hamilton**, 1843
  - $i^2 = j^2 = k^2 = ijk = -1$
  - usage in graphics until 1985 (Shoemake)
  - **generalization of complex numbers** in 4D space
- $\mathbf{q} = (\mathbf{v}, w) = i\,x + j\,y + k\,z + w = \mathbf{v} + w$
- **imaginary part** $\mathbf{v} = (x, y, z) = i\,x + j\,y + k\,z$
- $i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k$

Quaternions: operations I

- **addition**
  - $(\mathbf{v_1}, w_1) + (\mathbf{v_2}, w_2) = (\mathbf{v_1}+\mathbf{v_2}, w_1+w_2)$
- **multiplication**
  - $\mathbf{q}\,\mathbf{r} = (\mathbf{v_q} \times \mathbf{v_r} + w_r\mathbf{v_q} + w_q\mathbf{v_r},\ w_q w_r - \mathbf{v_q} \cdot \mathbf{v_r})$

$$\begin{gathered}
i(q_y r_z - q_z r_y + r_w q_x + q_w r_x),\\
j(q_z r_x - q_x r_z + r_w q_y + q_w r_y),\\
k(q_x r_y - q_y r_x + r_w q_z + q_w r_z),\\
q_w r_w - q_x r_x - q_y r_y - q_z r_z
\end{gathered}$$

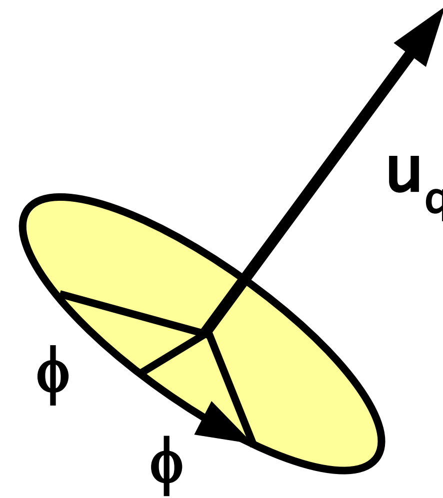
Quaternions: operations II

- **conjugation**
  - $(\mathbf{v}, w)^* = (-\mathbf{v}, w)$
- **norm** (squared absolute value)
  - $||\mathbf{q}||^2 = n(\mathbf{q}) = \mathbf{q}\,\mathbf{q}^* = x^2 + y^2 + z^2 + w^2$
- **unit**
  - $\mathbf{i} = (\mathbf{0}, 1)$
- **reciprocal**
  - $\mathbf{q}^{-1} = \mathbf{q}^* / n(\mathbf{q})$
- **multiplication by a scalar**
  - $s\,\mathbf{q} = (0, s)\,(\mathbf{v}, w) = (s\,\mathbf{v}, s\,w)$

Unit quaternions

- **unit quaternion** can be expressed by goniometry as
  - $\mathbf{q} = (\mathbf{u_q} \sin \phi, \cos \phi)$
  - for some **unit 3D vector** $\mathbf{u_q}$
- it represents a **rotation (orientation)** in 3D
  - ambiguity: both $\mathbf{q}$ and $\mathbf{-q}$ represent the same rotation!
  - identity (zero rotation): $(\mathbf{0}, 1)$
- power, exponential, logarithm:
  - $\mathbf{q} = \mathbf{u_q} \sin \phi + \cos \phi = \exp(\phi\, \mathbf{u_q})$, $\log \mathbf{q} = \phi\, \mathbf{u_q}$
  - $\mathbf{q}^t = (\mathbf{u_q} \sin \phi + \cos \phi)^t = \exp(t\phi\, \mathbf{u_q}) = \mathbf{u_q} \sin t\phi + \cos t\phi$

Rotation using quaternion



- **unit quaternion**
  - $\mathbf{q} = (\mathbf{u_q} \sin\phi, \cos\phi)$
  - $\mathbf{u_q}$ ... axis of rotation, $\boldsymbol{\phi}$ ... angle
- point or vector in 3D: $\mathbf{p} = [p_x, p_y, p_z, p_w]$
- **rotation** of point (vector) $\mathbf{p}$ around $\mathbf{u_q}$ by angle $\mathbf{2\phi}$
  - $\mathbf{p' = q\, p\, q^{-1} = q\, p\, q^*}$

Quaternion ↔ matrix conversions

- quaternion **q** converted to a matrix:

$$M = \begin{pmatrix} 1-2(y^2+z^2) & 2(xy+wz) & 2(xz-wy) \\ 2(xy-wz) & 1-2(x^2+z^2) & 2(yz+wx) \\ 2(xz+wy) & 2(yz-wx) & 1-2(x^2+y^2) \end{pmatrix}$$

- reverse conversion is based on equations ($):

$$m_{23} - m_{32} = 4wx$$
$$m_{31} - m_{13} = 4wy$$
$$m_{12} - m_{21} = 4wz$$
$$tr\, M + 1 = 4w^2$$

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Matrix → quaternion II

1. “matrix_trace+1” has large enough absolute value:

$$w=\frac{1}{2}\sqrt{tr\,M+1} \qquad x=\frac{m_{23}-m_{32}}{4w}$$

$$y=\frac{m_{31}-m_{13}}{4w} \qquad z=\frac{m_{12}-m_{21}}{4w}$$

2. otherwise compute a component with largest absolute value first and then apply $:

$$4x^2=1+m_{11}-m_{22}-m_{33}$$
$$4y^2=1-m_{11}+m_{22}-m_{33}$$
$$4z^2=1-m_{11}-m_{22}+m_{33}$$

Spherical linear interpolation (slerp)

- two quaternions **q** and **r** (**q** · **r** ≥ 0, else take **-q**)
- real parameter **0 ≤ t ≤ 1**

Interpolated quaternion slerp(q,r,t) = q (q* r)^{t}

- different formulation:

$$slerp(q,r,t) = \frac{\sin(\phi(1-t))}{\sin\phi} \cdot q + \frac{\sin(\phi t)}{\sin\phi} \cdot r$$

$$\cos\phi = q_x r_x + q_y r_y + q_z r_z + q_w r_w$$

… the shortest spherical arc between **q** and **r**

Rotation between two vectors

- two vectors **s** and **t**:

1. normalization of **s**, **t**
2. unit rotation axis $\mathbf{u} = (\mathbf{s} \times \mathbf{t}) \,/\, ||\mathbf{s} \times \mathbf{t}||$
3. angle between **s** and **t**: $\mathbf{e} = \mathbf{s} \cdot \mathbf{t} = \cos 2\phi$

   $||\mathbf{s} \times \mathbf{t}|| = \sin 2\phi$
4. final quaternion: $\mathbf{q} = (\,\mathbf{u} \cdot \sin\phi, \cos\phi\,)$

$$q = (q_v, q_w) = \left(\frac{1}{\sqrt{2(1+e)}} (s \times t), \frac{\sqrt{2(1+e)}}{2} \right)$$

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Slerp of rotational matrices

- two rotational matrices **Q** and **R**
- real parameter $\mathbf{0 \le t \le 1}$

Interpolated matrix $\mathbf{slerp(Q,R,t) = Q\,(Q^T R)^t}$

- **technical problem:** ? power operation on matrices
- need to compute axis and angle $Q^T R$
  - not very efficient

(for details see “RotationIssues.pdf”)

Rotation representation – summary

- **rotational matrix**

  \+ HW support, efficient point/vector transformation

  – memory (float[9]), other operations are not so efficient

- **rotational axis and angle**

  \+ memory (float[4] or float[6]), similar to quaternion

  – inefficient composition and interpolation

- **quaternion**

  \+ memory (float[4]), composition, interpolation

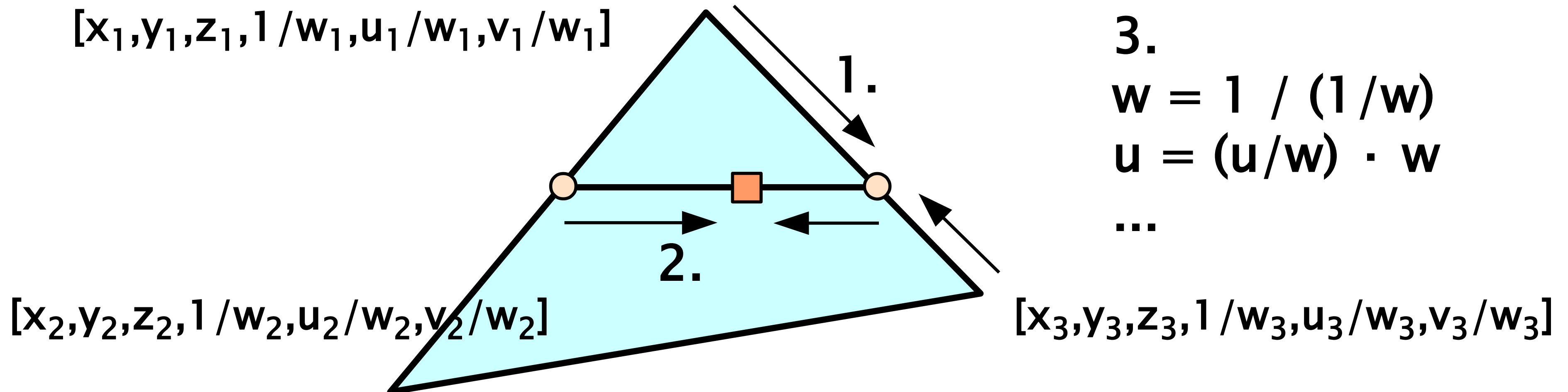
  – inefficient point/vector transformation

(for details see “RotationIssues.pdf”)

Interpolation in the screen space

- concerns **clip-space** and next spaces
  - clip space: [x, y, z, w]
  - NDS: [x/w, y/w, z/w, w] (“w” could be preserved)
  - window space (fragments): [x_i, y_i, z_i, w]
- **projective perspective transformation** maps depth **z** to NDS **non-linearly !**
  – **nonuniform accuracy** of z-buffer (distant parts could be less accurate: minimization of **f/n** ratio !)
  + **interpolation of depth z** can be done in the screen space **linearly**
- **W-buffer** instead of z-bufferu (not used any more)

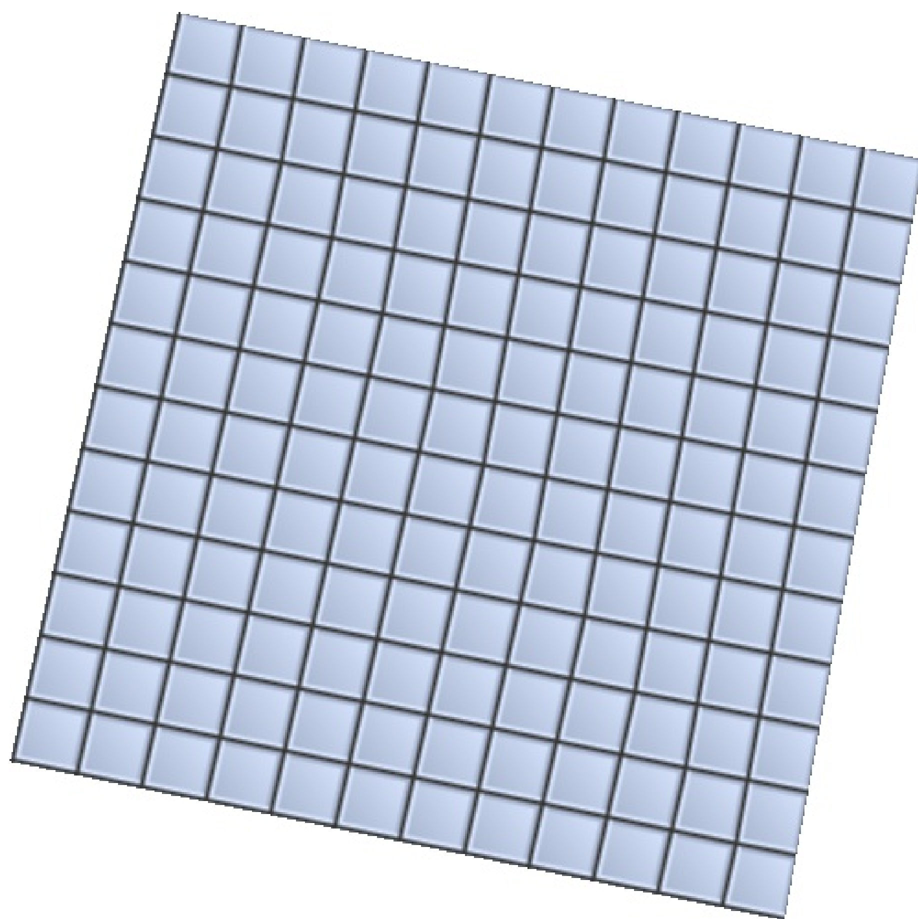
Perspective-correct interpolation



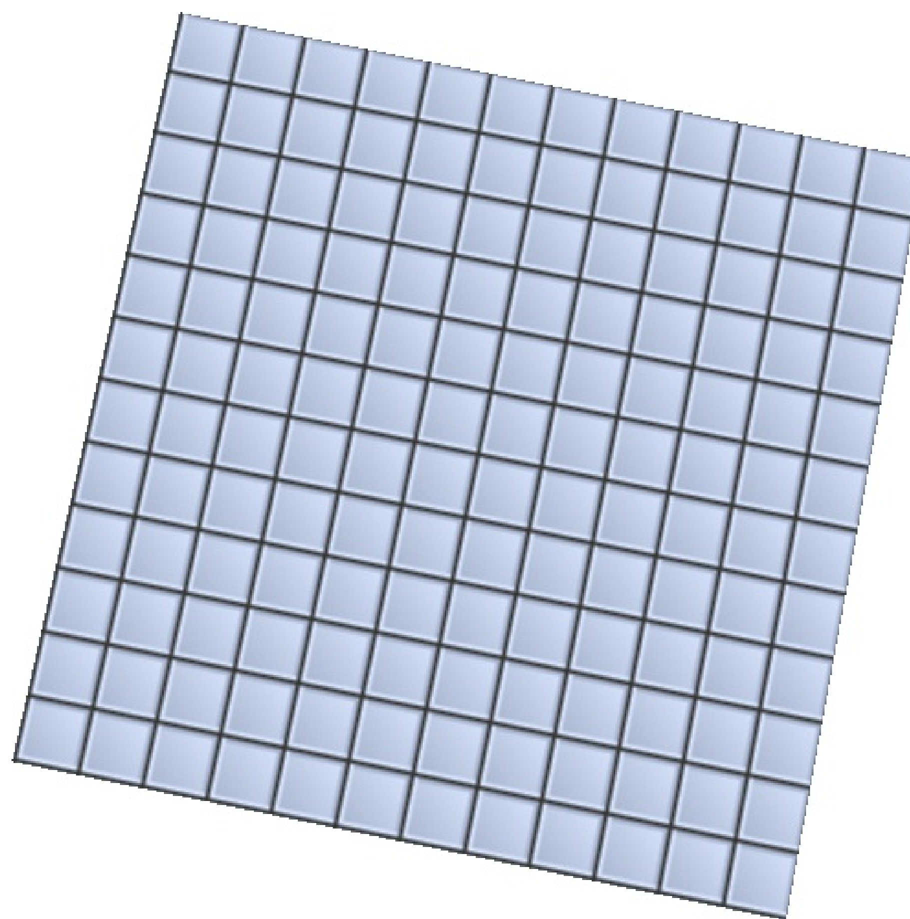
Linear interpolation + division:

- **depth** "**z**" is linear by itself
- for **texture coordinates** or "**w**" perspective-correct interpolation must be used:
- quantities "**1/w**", "**u/w**", "**v/w**", .. are linearly interpolated, **[u, v]** is then computed by division (**hyperbolic interpolation**)

Interpolation example I

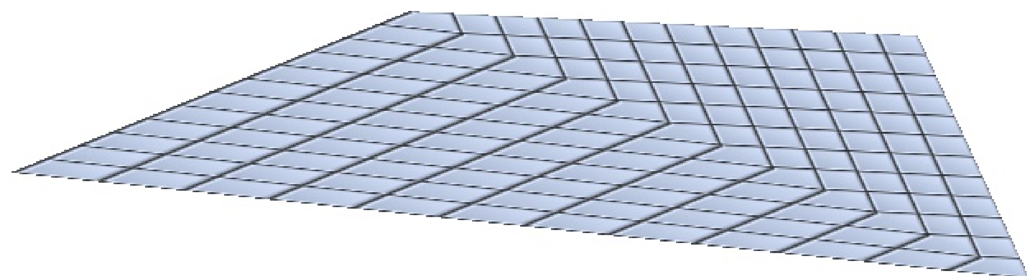


Affine

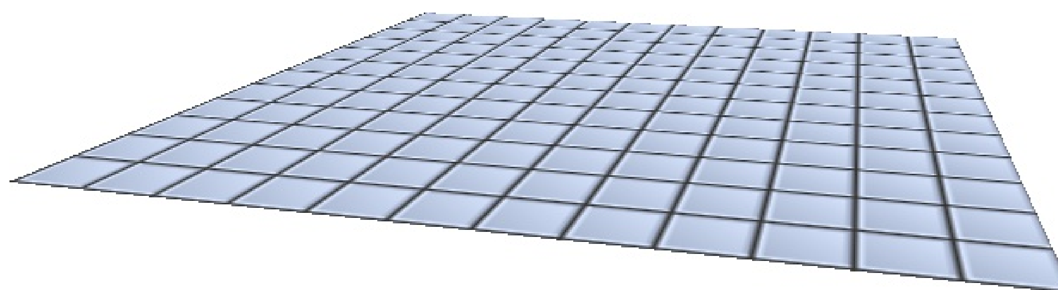


Correct

Interpolation example II

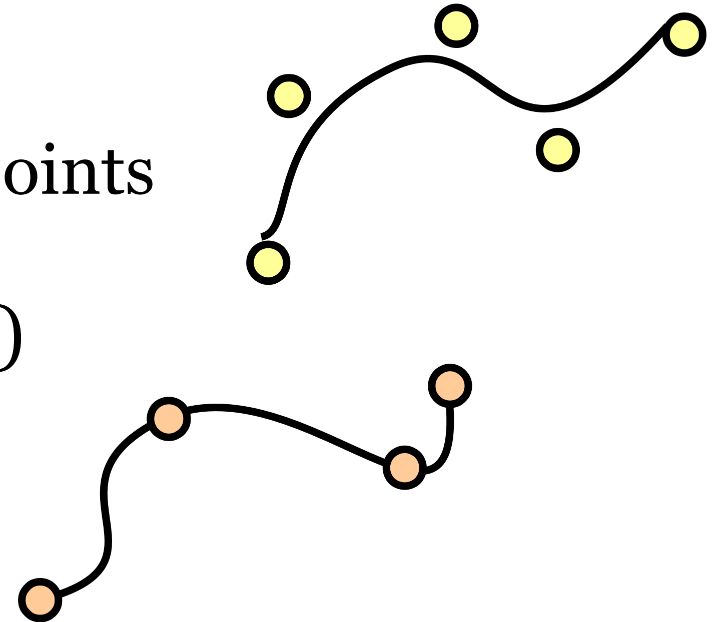


Affine



Correct

Approximation and interpolation



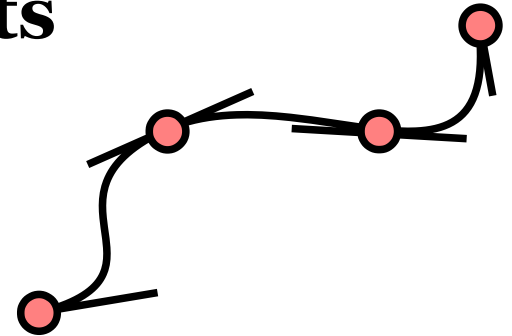
- **approximation** (e.g. B-spline)
  - needs not to pass through control points
- **interpolation** (e.g. Catmull-Rom)
  - curve passes through ctrl. points
- **curve continuity**
  - $\mathbf{G^n}$ – geometric continuity of nth order ($\mathbf{G^0}$ – simple continuity, $\mathbf{G^1}$ – tangent, $\mathbf{G^2}$ – curvature, ...)
  - $\mathbf{C^n}$ – analytical continuity of nth order, n^{th} derivative cont. ($\mathbf{C^1}$ – speed, $\mathbf{C^2}$ – acceleration), superior to geometric

History

- **curves in modeling industry**
  - Paul de Faget **de Casteljau** by Citroën (1959)
  - Pierre **Bèzier** (by Renault 1933-1975, UNISURF)
    - late start, but his results were more popular
  - application of spline function theory – mostly in the USA (James **Ferguson**, 1964, Boeing, C^2 spline curves)
- **spline function theory**
  - B-spline: Isaac Jacob **Schoenberg**, (ballistics, Ab-erdeen, MD, 1946)
  - theory: Carl **de Boor** (also worked in General Motors)
  - Gordon, Riesenfeld united Bèzier and B-spline curves (1972)

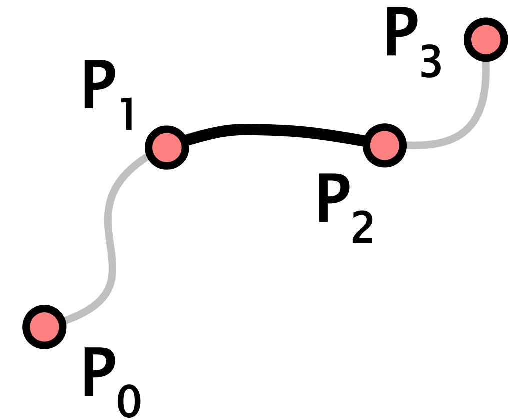
“Free–form“ curves I

- defined by a sequence of **control points**
  - “control polygon”
  - approximation or interpolation
  - boundary conditions can be different



- **controllability**
  - sometimes tangent vectors added in ctrl. pts. (Hermit)
  - interpolation ⇒ closer control
- **locality**
  - change of single control point (one tangent vector) induces change in restricted neighborhood only

“Free-form“ curves II



- **parametric** expression (**0 ≤ t ≤ 1**)

$$P(t) = \sum_{i=0}^{N-1} w_i(t)\, P_i$$

- **convex hull** property
  - curve lies in convex hull of its control polygon
- **Cauchy condition** for blending functions
  - sufficient for <u>convex hull property</u>
  - ensures <u>affine transformation invariancy</u>

$$\sum_{i=0}^{N-1} w_i(t) = 1$$

Splines

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Spline functions

Named after elastic ruler used in ship design (pinned in several points by “ducks”)

- definition: **spline function of degree *n***:
  - piece-wise polynomial (of degree ***n***)
  - maximum-smoothness connection:

    $\mathbf{C^{n-1}}$ – continuity of $\boldsymbol{n\text{-}1}^{th}$ derivative (polyn. of degree ***n***)
  - global parametrization $\mathbf{u}$, $\mathbf{u_0 \le u \le u_N}$ $[u_0, u_1, \dots u_N]$
  - individual parts are often uniformly parametrized – uniform spline: $\mathbf{t_i = (u - u_i) / (u_{i+1} - u_i)}$, $\mathbf{0 \le t_i \le 1}$

Polynomial curve

- matrix notation:

$$P(t) = \boldsymbol{TC} = [t^n, t^{n-1}, \dots t, 1] \cdot \begin{bmatrix} x_n & y_n & z_n \\ x_{n-1} & y_{n-1} & z_{n-1} \\ .. & .. & .. \\ x_1 & y_1 & z_1 \\ x_0 & y_0 & z_0 \end{bmatrix}$$

- basis matrix **M** and vector of geometric conditions **G**:

$$\boldsymbol{C} = \boldsymbol{MG} = [m_{ij}]_{i=n,\, j=1}^{0,\, k} \cdot \begin{bmatrix} G_1 \\ .. \\ G_k \end{bmatrix} \qquad P(t) = \boldsymbol{TMG}$$

Matrix notation of a curve

- **P(t) = T C = T M G**
  - separation of a parameter vector (**T**) from polynomial basis (**M**) and geometric control conditions/points (**G**)
  - differentiation (tangent, curvature) restricted to **T**
  - control polynomial **TM** times “geometry” **G**

- **cubic**: n = 3, k = 4

$$Q(t)=[t^3, t^2, t, 1]\cdot\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}\cdot\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}$$

Hermite cubic curve

- **Ferguson curve** (cubic)
- geometry: **endpoints** and **tangent vectors**
  - beginning ($\mathbf{P_0}$) and end ($\mathbf{P_1}$) of a curve
  - tangents in beginning ($\mathbf{T_0}$) and ending ($\mathbf{T_1}$) points

$$F(t) = [t^3, t^2, t, 1] \cdot \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ T_0 \\ T_1 \end{bmatrix}$$

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Hermite cubic – examples

T_0 P_0 P_1 T_1

T_0 P_0 P_1 T_1

T_0 P_0 P_1 T_1

More curves

- **interpolating cubics** derived from Hermite:
  - general: **Kochanek-Bartels** (KB-spline, TCB cubic)
  - special: **cardinal** spline, **Catmull-Rom** spline
  - **Akima** interpolation (“Akima spline”, not C^2)
  - **D-spline** cubic
- another popular curves:
  - **Bèzier** curves
  - **B-spline** curve, **Coons** spline (approximation)
  - **natural** spline (interpolation)

Kochanek–Bartels cubic (KB–spline, TCB)

- derived from **Hermite** cubic (3DS Max, Lightwave)
  - tangent vectors are derived from control points
  - three additional scalar parameters (zero by default):
    - “**tension**” **t**: sharpness of a curve passing control point (absolute value of a tangent vector)
    - “**continuity**” **c**: in control points
    - “**bias**” **b**: tangent direction in control point

Left and **right** tangent ($\mathbf{T_0}$ and $\mathbf{T_1}$ in local sense):

$$L_i = \frac{(1-t)(1-c)(1+b)}{2} \cdot (P_i - P_{i-1}) + \frac{(1-t)(1+c)(1-b)}{2} \cdot (P_{i+1} - P_i)$$

$$R_i = \frac{(1-t)(1+c)(1+b)}{2} \cdot (P_i - P_{i-1}) + \frac{(1-t)(1-c)(1-b)}{2} \cdot (P_{i+1} - P_i)$$

Cardinal spline, Catmull–Rom spline

Special cases of KB-spline:

- **cardinal spline**
  - parameter **a** only (in fact relates to “**t**”, **c = b = o**)

$$T_i = a \cdot (P_{i+1} - P_{i-1}) \qquad 0 \leqslant a \leqslant 1$$

- **Catmull-Rom spline**
  - a = t = 1/2

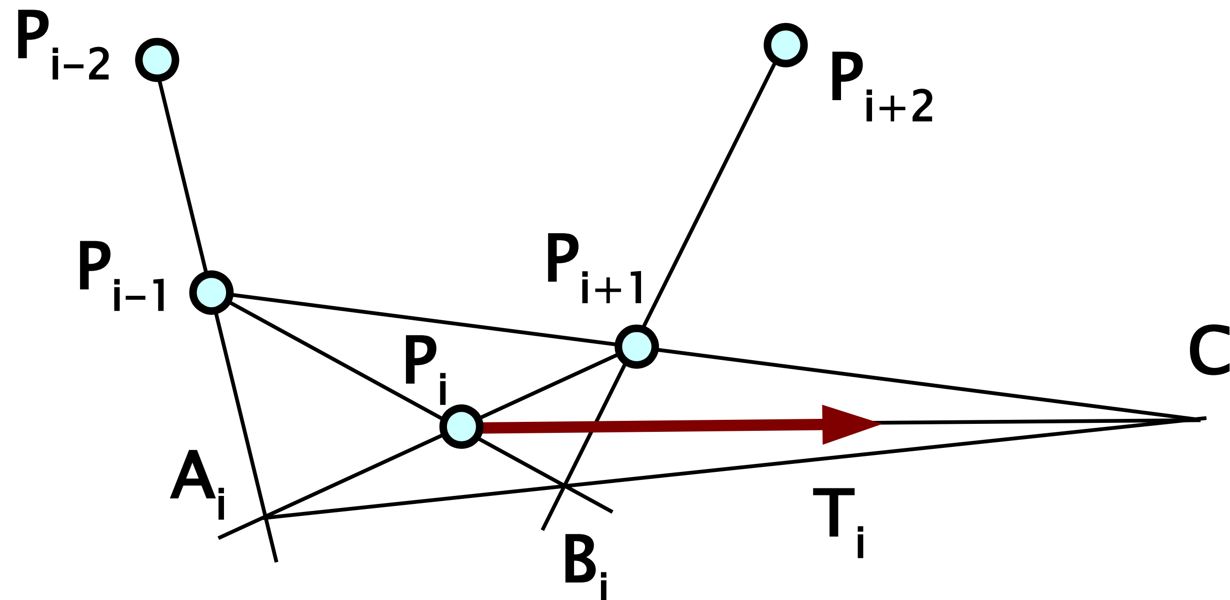
$$T_i = \frac{1}{2} \cdot (P_{i+1} - P_{i-1})$$

$$\boldsymbol{MG} = \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

Akima interpolation

Alternative definition of tangent vectors for Hermite cubic:

- non-C² !



$$|T_i| = |P_{i+1} - P_{i-1}|$$

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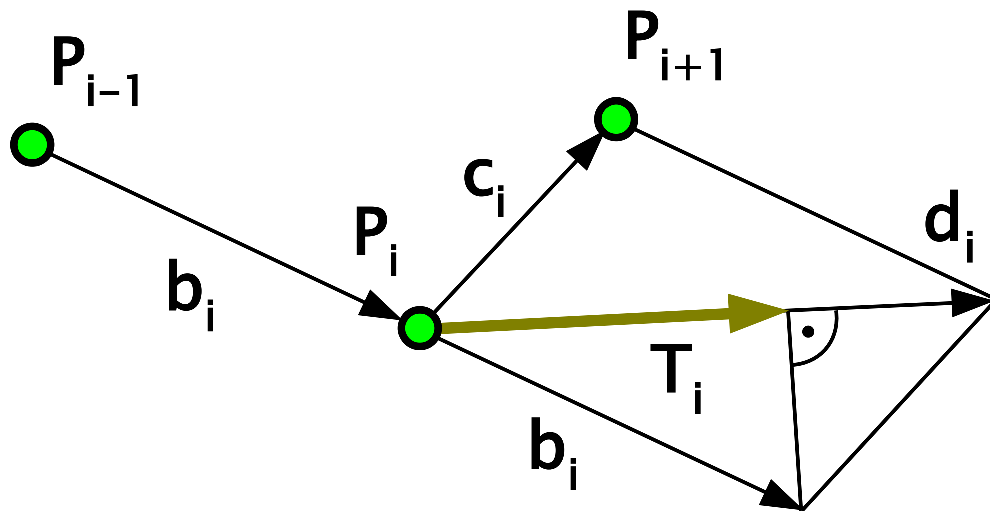
D–spline cubic

- one more variant of Hermite cubic
  - tangent vector computed by the **“D-interpolation”**



$$G = \begin{bmatrix} P_i \\ P_{i+1} \\ T_i \\ T_{i+1} \end{bmatrix}$$

Bèzier curves I

- **polynomial curve of degree N**
  - **N+1** control points
    - boundary control points define endpoints of a curve
    - boundary control-point pairs define tangent vectors
  - parametric expression using Bernstein polynomials
  - easy $\mathbf{G^1}$ or $\mathbf{C^1}$ connection
  - spline-join is also possible, but much more complicated

Bernstein polynomials:

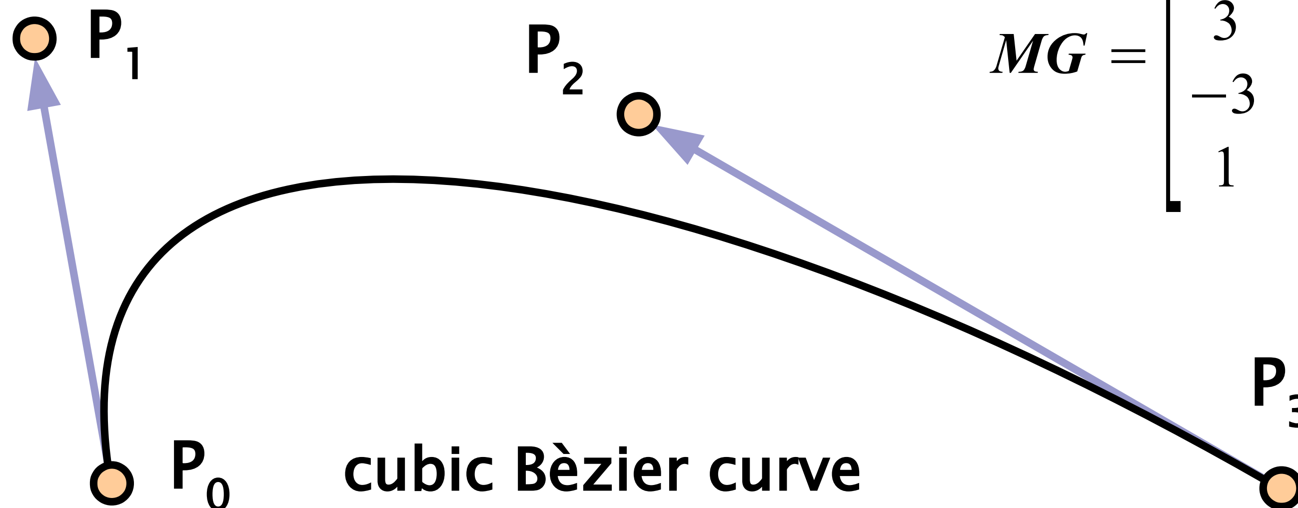
$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad 0 \leqslant i \leqslant n, \quad 0 \leqslant t \leqslant 1$$

Bèzier curves II

Cauchy condition

⇒ convex combination of control points

$$\sum_{i=0}^{n} B_i^n(t) = 1 \quad \mathit{for} \quad 0 \leqslant t \leqslant 1$$



$$\boldsymbol{MG} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Joining Bèzier curves I

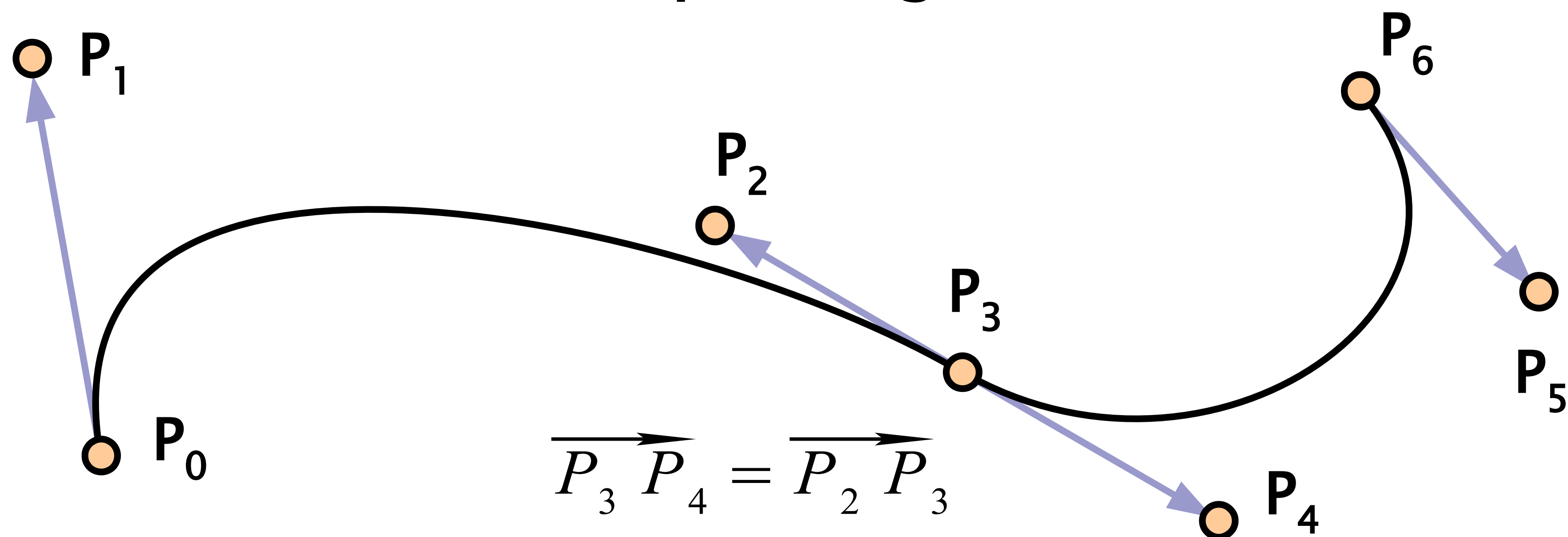
G^1 connection (co-linear tangents)

P_1 P_2 P_6 P_5 P_3 P_0 P_4

$$\overrightarrow{P_3 P_4} = k \cdot \overrightarrow{P_2 P_3}$$

Joining Bèzier curves II

C[1] connection (equal tangent vectors)



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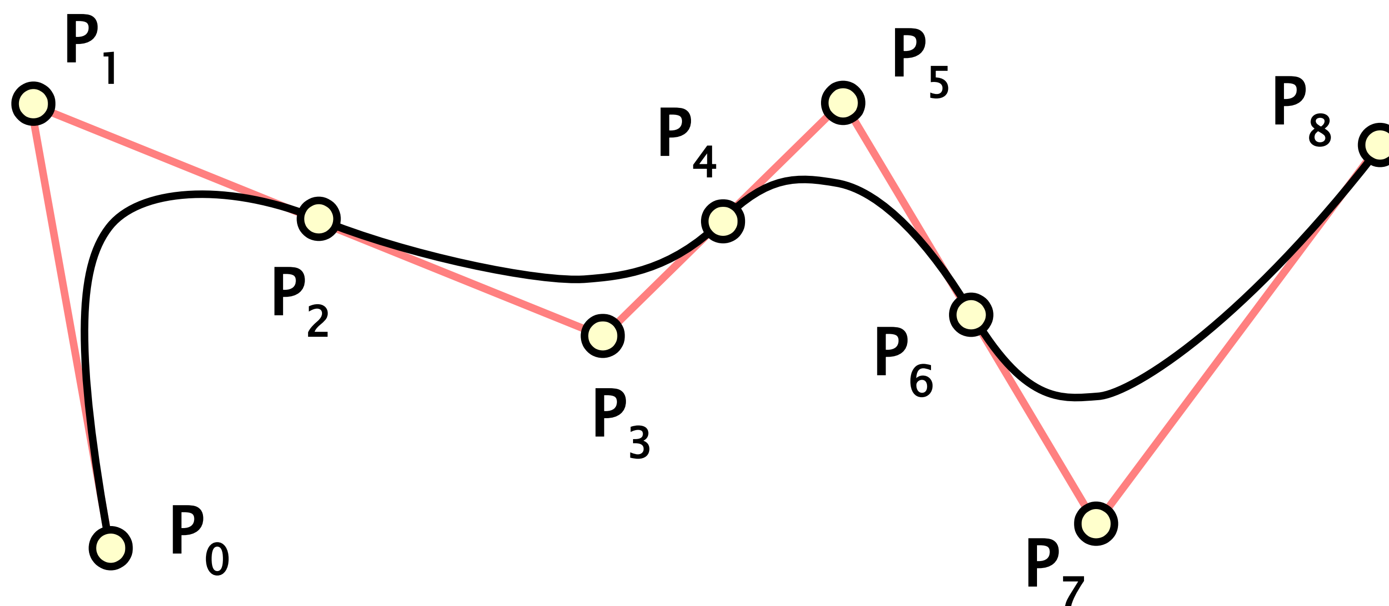
Joining Bèzier curves III

Quadratic spline from Bèzier segments



$$\overrightarrow{P_1 P_2} = \overrightarrow{P_2 P_3} \quad \overrightarrow{P_3 P_4} = \overrightarrow{P_4 P_5} \quad \ldots \quad \overrightarrow{P_{2k-1} P_{2k}} = \overrightarrow{P_{2k} P_{2k+1}}$$

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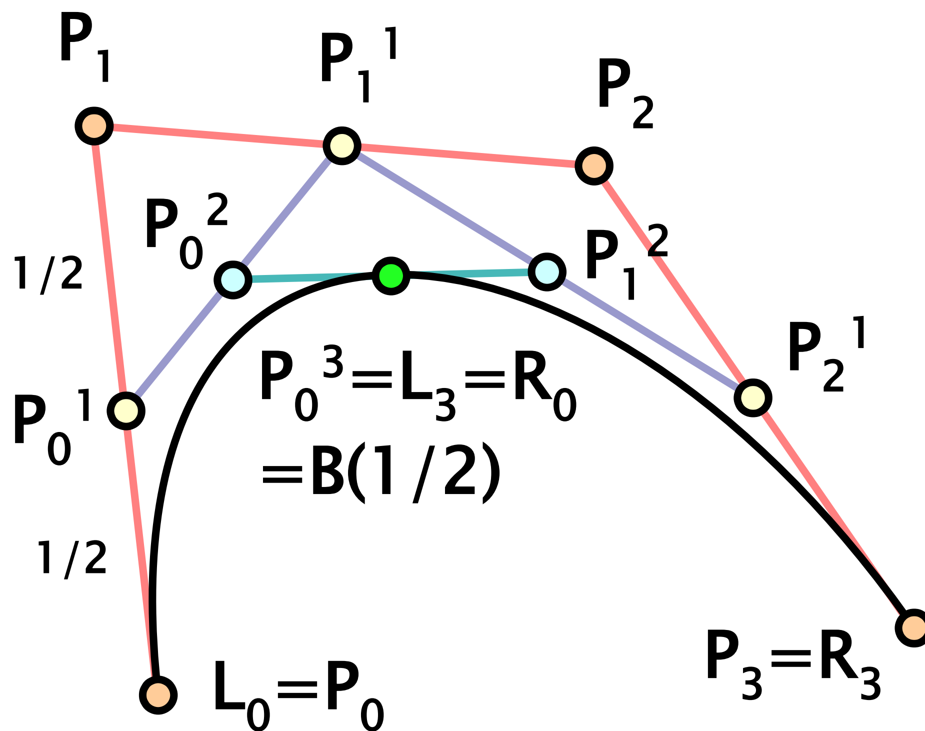
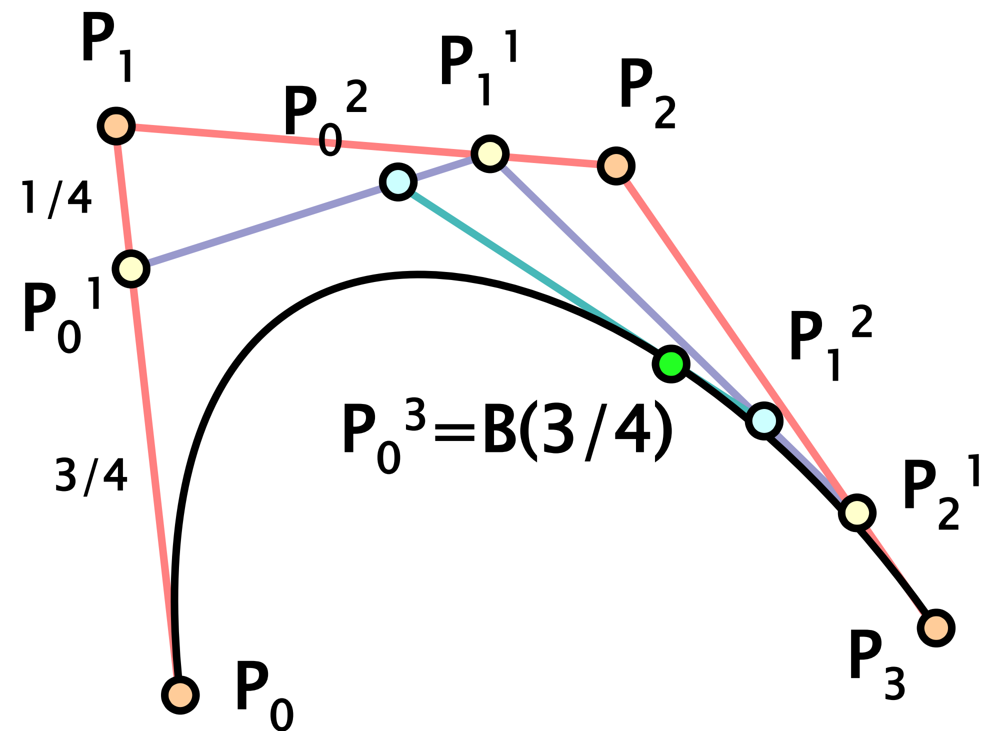
De Casteljau (de Boor) algorithm

- **geometric construction** of Bèzier curve
  - used as “**subdivision**” scheme or for computation of a specific point..





Cubic spline

- function assembled from **cubic polynomials**
  - neighbor polynomials have C^2 joint
  - elastic “spline-ruler” (see construction)
- **interpolating cubic spline**
  - in <u>knot points</u> $\mathbf{x_0}, \mathbf{x_1}, \dots \mathbf{x_n}$ <u>function values</u> $\mathbf{y_0}, \mathbf{y_1}, \dots \mathbf{y_n}$ are prescribed

$$S(x) = S_k(x) = s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3$$
$$x \in [x_k, x_{k+1}], \quad k = 0, 1, \dots, n-1$$

- condition **A**: $S(x_k) = y_k \qquad k = 0, 1, \dots, n$

Interpolating cubic spline

- condition **B** (C^0 continuity):

$$S_k(x_{k+1}) = S_{k+1}(x_{k+1}) \qquad k=0,\,1,\,\ldots,\,n-2$$

- condition **C** (C^1 continuity):

$$S'_k(x_{k+1}) = S'_{k+1}(x_{k+1}) \qquad k=0,\,1,\,\ldots,\,n-2$$

- condition **D** (C^2 continuity):

$$S''_k(x_{k+1}) = S''_{k+1}(x_{k+1}) \qquad k=0,\,1,\,\ldots,\,n-2$$

- **natural cubic spline** has additional condition **E**:

$$S''(x_0) = S''(x_n) = 0$$

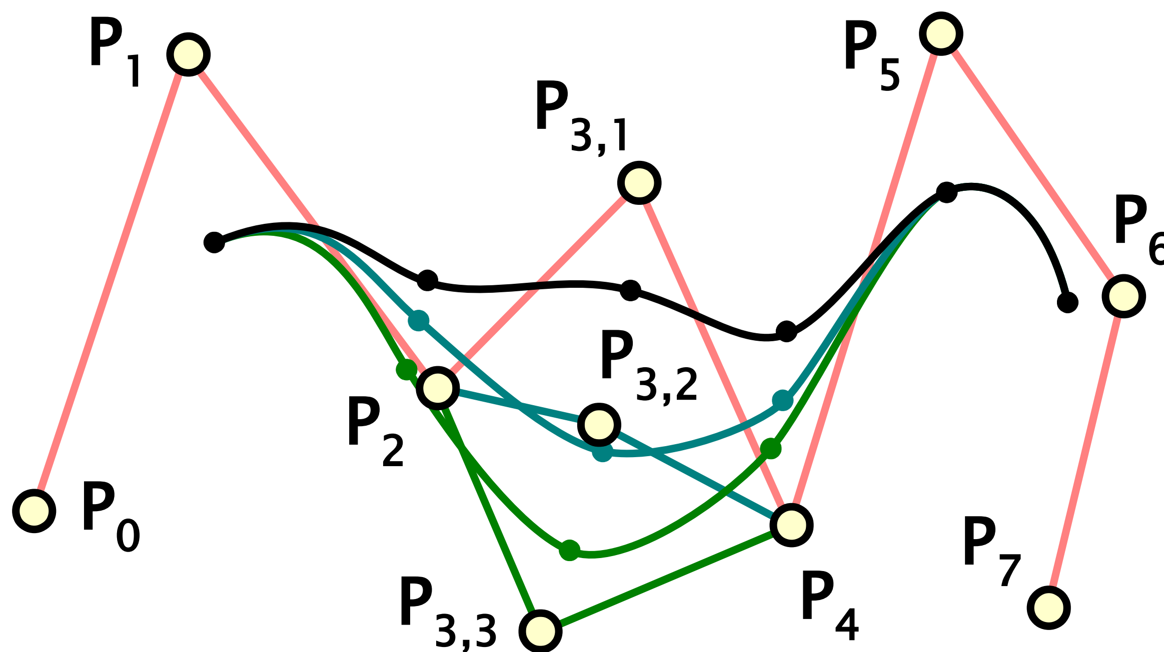
Natural cubic spline

- **interpolating spline**
  - uniquely determined by the conditions (solution of linear system of equations $\mathbf{s}_{k,l}$)
  - has no local property (the whole curve changes after altering one control point)
- **open spline**:
  - conditions **A**, **B**, **C**, **D** are not sufficient, 2 more DoF
  - additional condition **E** (second derivatives in endpoints)
- **closed** (cyclic) **spline**: $\mathbf{x_0} = \mathbf{x_n}$
  - **C** and **D** give us missing conditions for $\mathbf{x_0}$

B–spline (basis spline)

- **“free-form”** curve
  - shape is defined by a sequence of **control points**
  - parametric form using **basis/blending functions** (dependency of a curve point on control polygon)
  - **local property** (only local change after altering one CP)
- **uniform cubic B-spline** (Coons curve)
  - unified set of basis functions (cubic polynomials)
- **nonuniform B-spline**
  - more complicated definition using knot vector $\mathbf{0 \le t_i \le 1}$

Coons B-spline

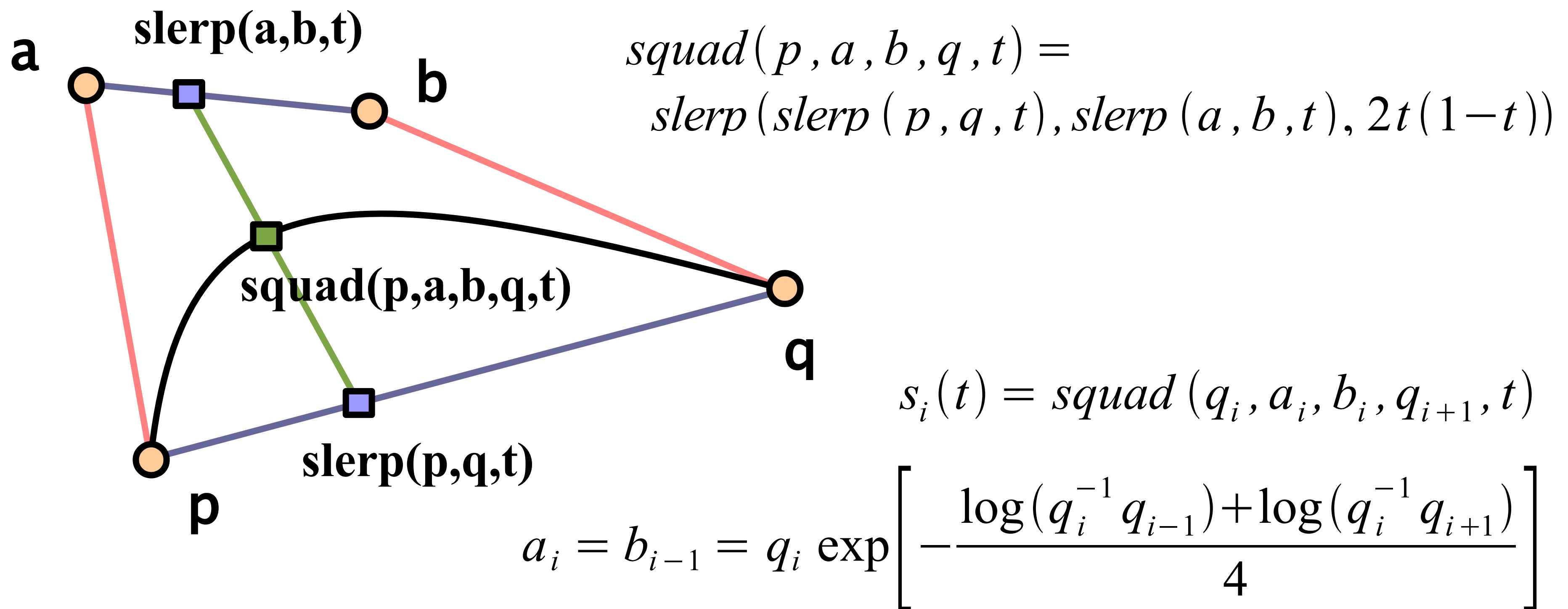


- continuity **C²**
- <u>sharing</u> 3 CP between neighb.
- altering one CP induces change in <u>4 segments</u>

$$\boldsymbol{MG} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

Spline interpolation of quaternions

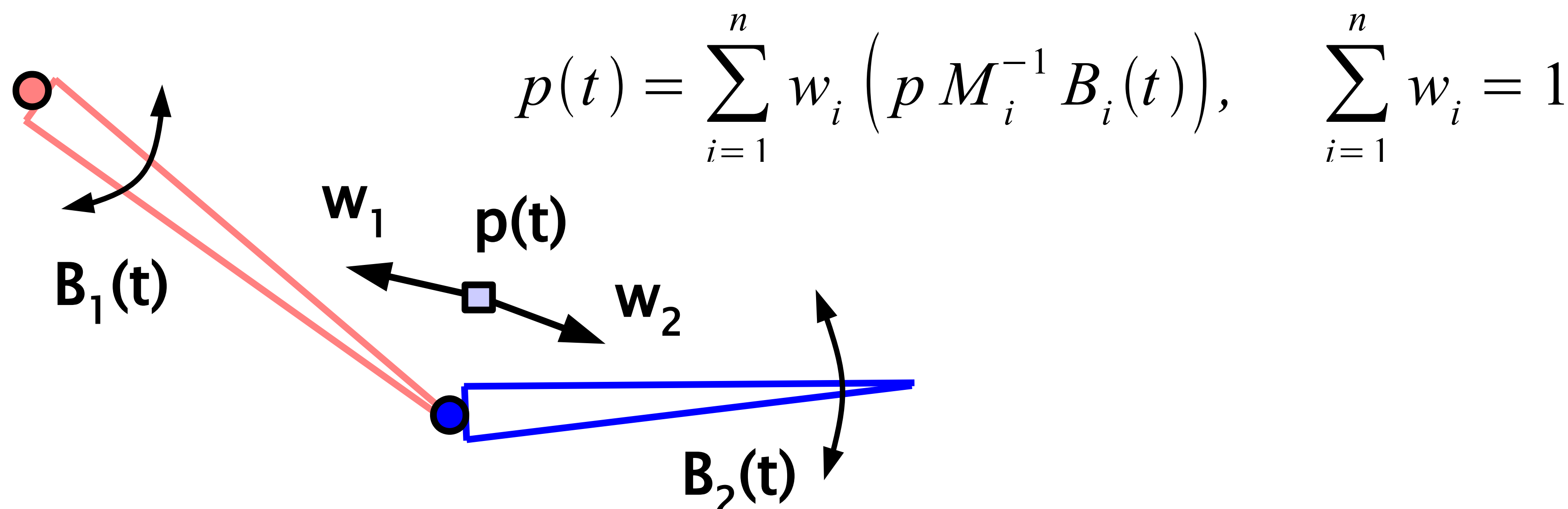
- subsequent interpolation by a sequence of orianta-tions $\mathbf{q_0}, \mathbf{q_1}, \dots \mathbf{q_n}$
  - $\mathbf{slerp(q_i,q_{i+1},t)}$ not sufficient continuity (C^0 only)



Vertex blending (“skinning”)

- **vertex p** of each triangle is connected to some neighbor “**bones**” (coordinate systems)
  - quiescent position: t = 0
  - bones are **animated** by matrices $\mathbf{B_i(t)}$ (object → world)
  - initial transformation $\mathbf{M_i = B_i(0)}$

$$p(t) = \sum_{i=1}^{n} w_i \left(p\, M_i^{-1} B_i(t) \right), \qquad \sum_{i=1}^{n} w_i = 1$$



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