Ray × scene intersections

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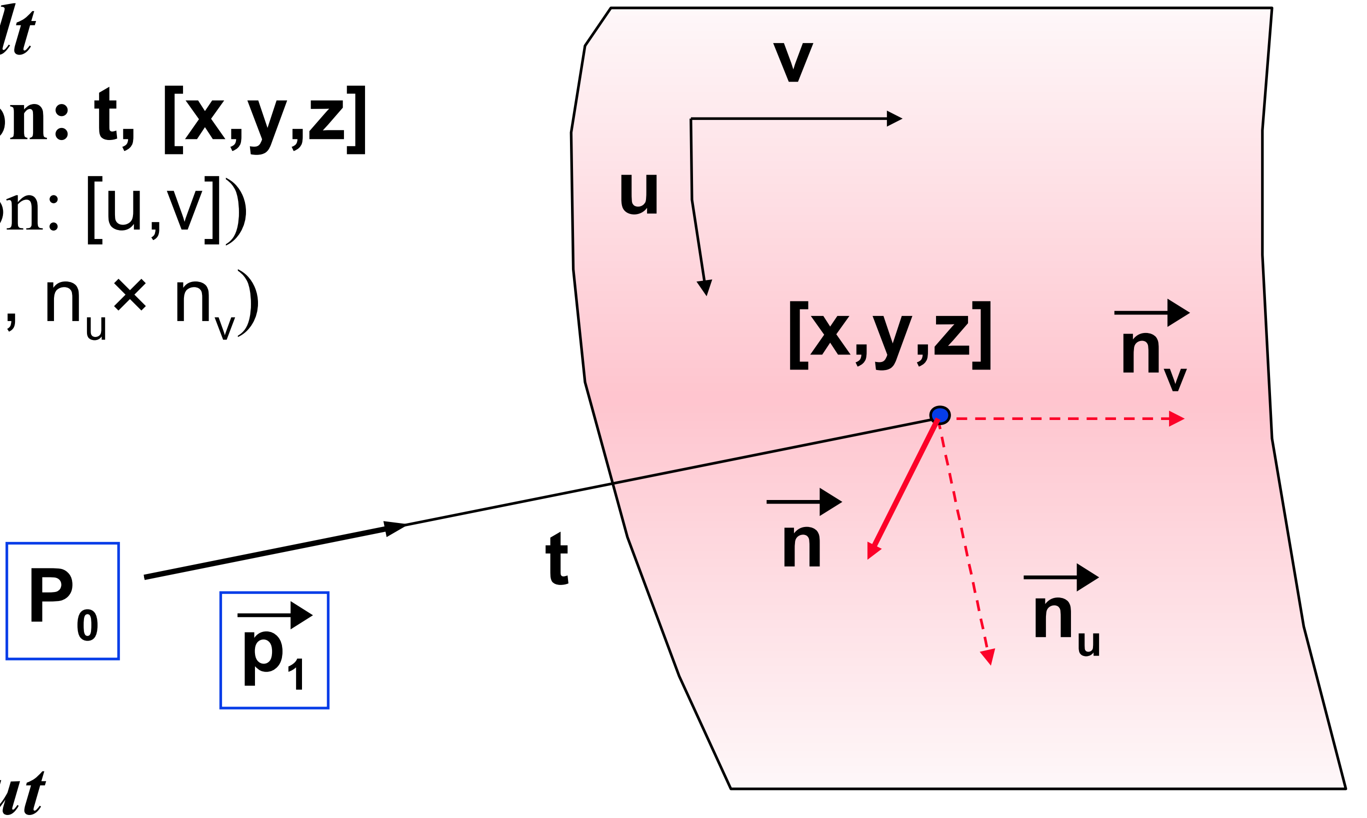
Ray × scene intersection

result

3D position: t, [x,y,z]

(2D position: [u,v])

(Normal: n, $n_u \times n_v$)



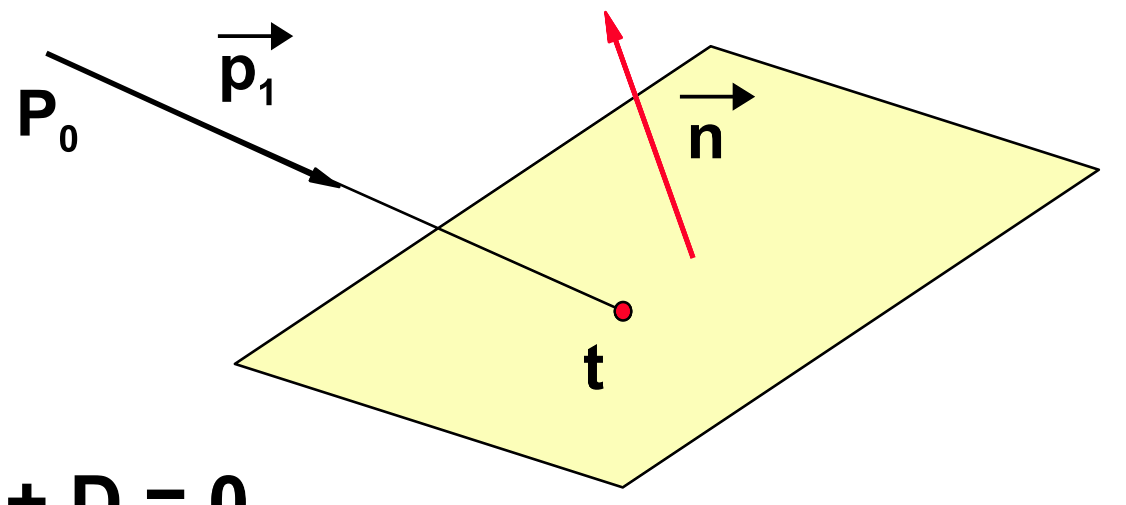
input

Ray: P_0, $\vec{p}_1$

Plane

ray:

$$P(t) = P_0 + t \cdot \vec{p_1}$$



plane:

$$n = [x_n, y_n, z_n]$$

$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$$

- intersection $t = -(n \cdot P_0 + D) / (n \cdot p_1)$
- negative: **2**±, **3***, positive: **5**±, **6***, **1**/
- computation of **[x,y,z]**: **3**±, **3***

Inverse transformation on the plane

plane:

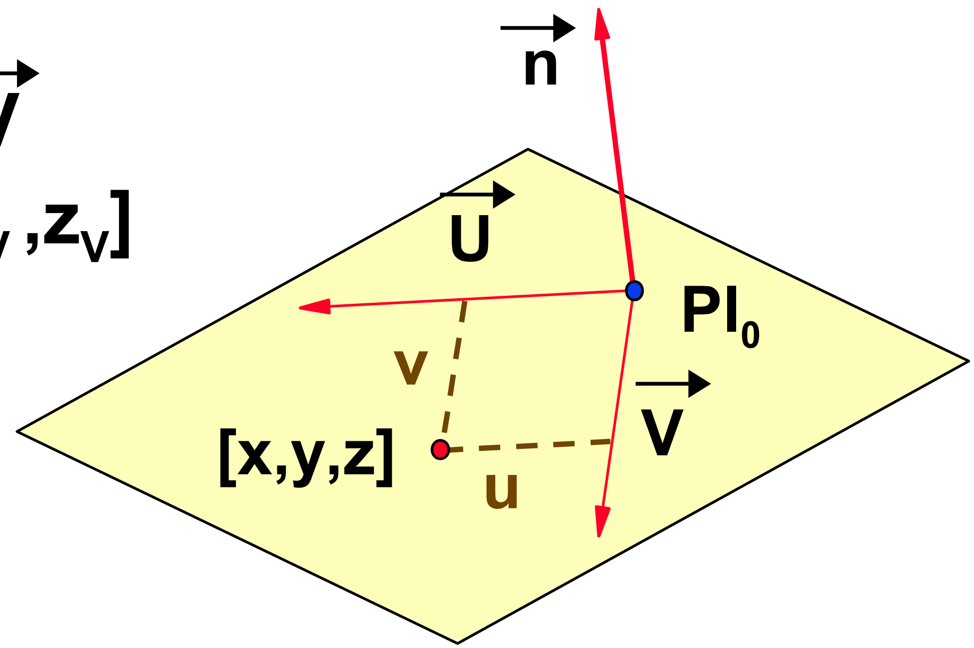
$Pl(u,v) = Pl_0 + u \cdot \vec{U} + v \cdot \vec{V}$

$\vec{U} = [x_U, y_U, z_U]$, $\vec{V} = [x_V, y_V, z_V]$

$\vec{n} = \vec{U} \times \vec{V}$

input: Pl, $\vec{U}$, $\vec{V}$, $[x,y,z]$

result: $[u,v]$



- linear system $u \cdot x_u + v \cdot x_v = x - Pl_{0x}$

  $u \cdot y_u + v \cdot y_v = y - Pl_{0y}$

- solution **[u,v]**: **5**±, **5***, **2**/

Parallelogram

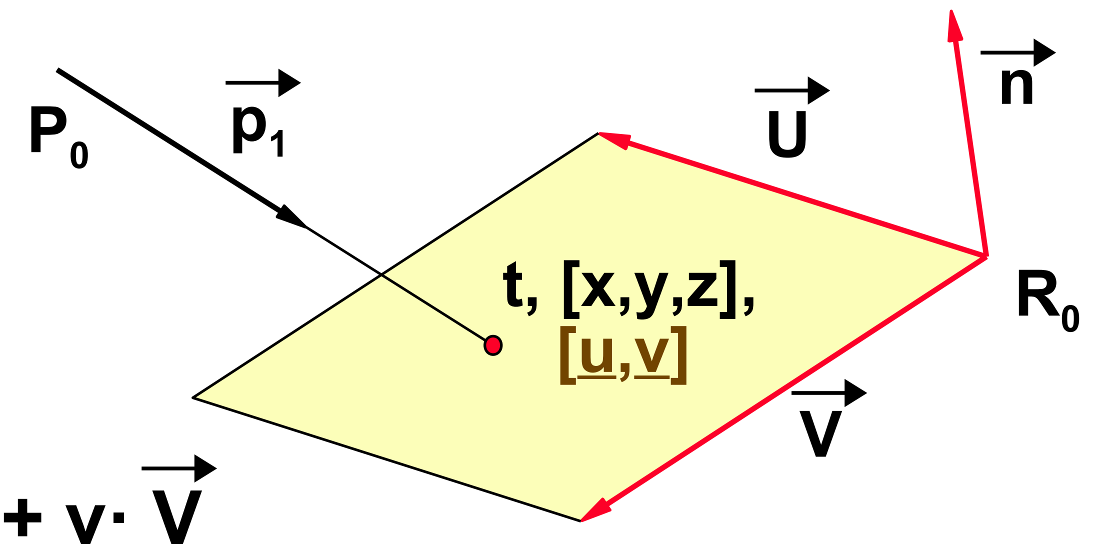
ray:

$P(t) = P_0 + t \cdot \vec{p_1}$

parallelogram:

$R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}$

$0 \le u,v \le 1$

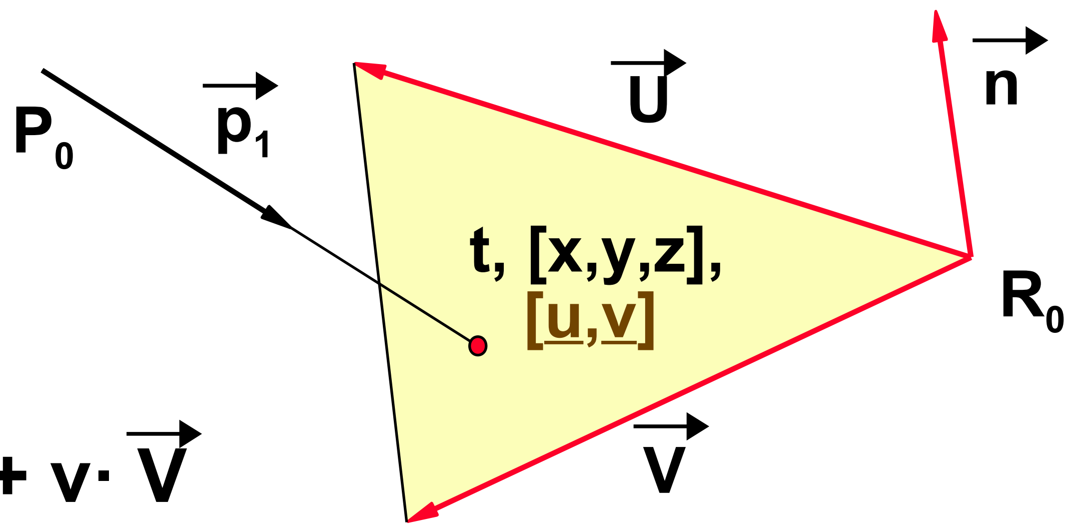


- computing **t**, **[x,y,z]**, **[u,v]**, tests of **u,v**
- positive case total: **13±, 14*, 3/, 4≤**

Triangle

ray:

$$\mathbf{P(t) = P_0 + t \cdot \vec{p}_1}$$



triangle:

$$\mathbf{R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}}$$

$$\mathbf{0 \le u, v, \underline{u+v} \le 1}$$

- computing **t**, **[x,y,z]**, **[u,v]**, tests of **u,v**
- positive case total: **14±**, **14***, **3/**, **3≤**

General planar polygon

ray:

$P(t) = P_0 + t \cdot \vec{p_1}$

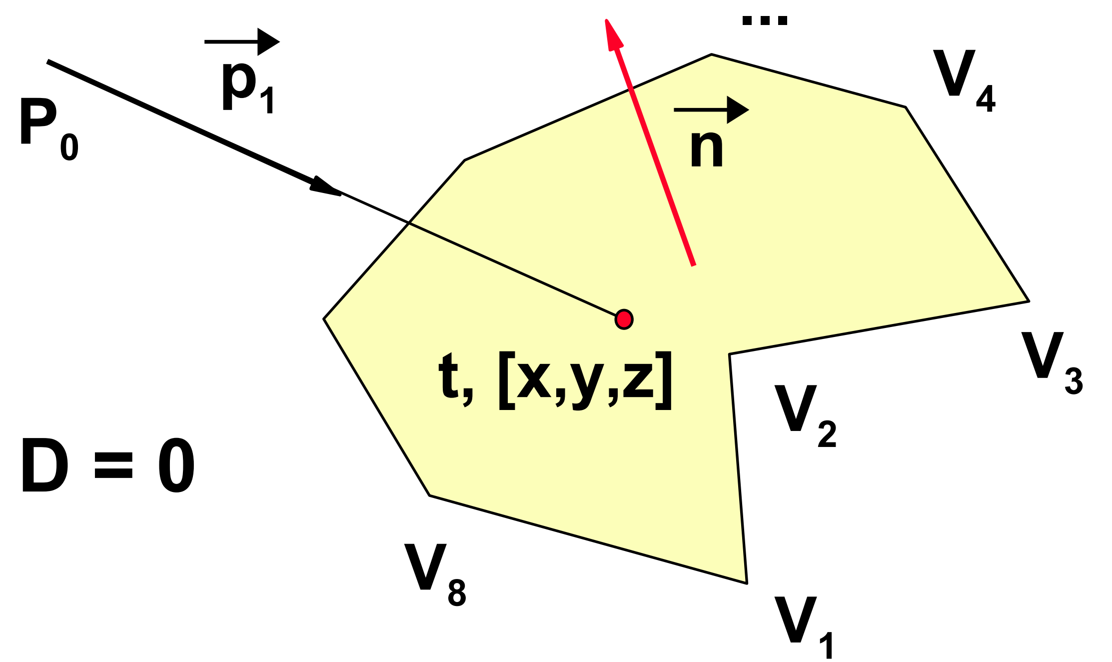
polygon plane:

$\vec{n} = [x_n, y_n, z_n]$

$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$

polygon vertices:

$V_1, V_2, \ldots V_M$

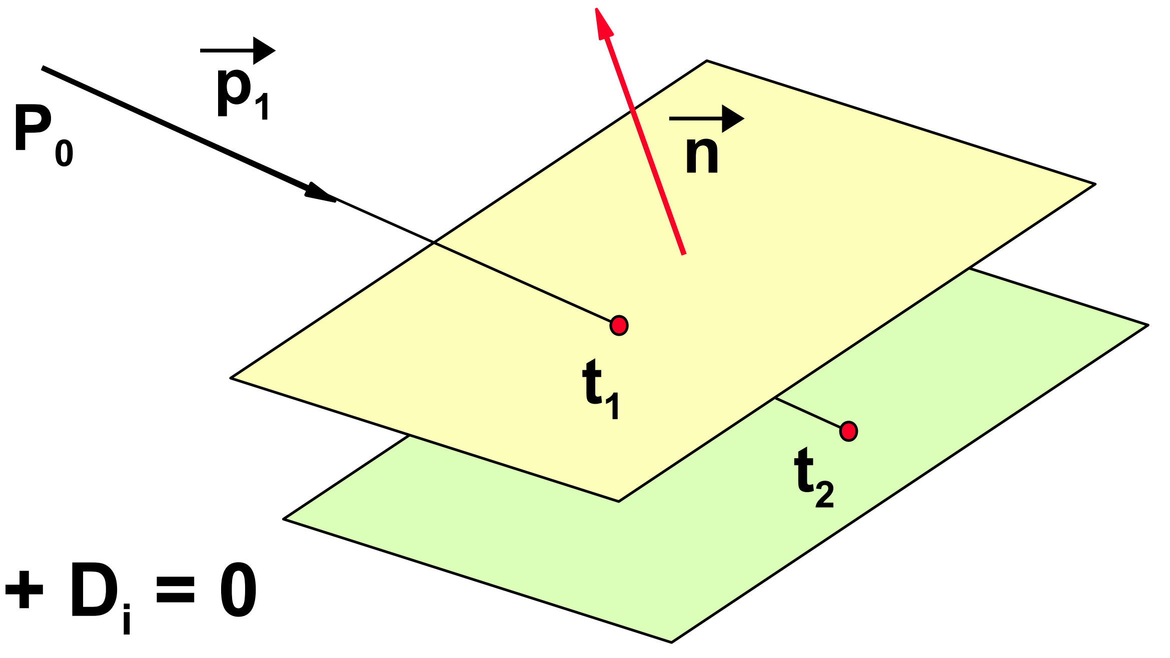


- computing **t, [x,y,z]**, planar test: **point × polygon**
- intersection with the plane: **8±, 9*, 1/**

Parallel planes

ray:

$P(t) = P_0 + t \cdot \vec{p_1}$



parallel planes:

$\vec{n} = [x_n, y_n, z_n]$

$x \cdot x_n + y \cdot y_n + z \cdot z_n + D_i = 0$

- intersections $t_i = -(n \cdot P_0 + D_i) / (n \cdot p_1)$
- the 1st plane: **5**±, **6***, **1**/, every next one: **1**±, **1**/

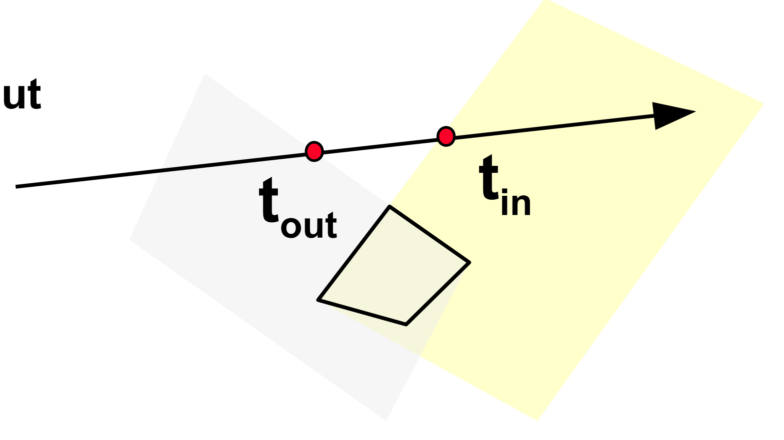
Convex polyhedron

- defined as an **intersection of K halfspaces**
  - at most **K** intersections ray vs. plane
  - **parallelism** of planes can be used – e.g. cuboid

- variables $\mathbf{t_{in}}$, $\mathbf{t_{out}}$ initialized to **0**, **∞**
- ray vs. one halfspace: $\langle$ **t**, **∞**) resp. (**–∞**, **t** $\rangle$

  $\mathbf{t_{in} = max\{ t_{in} , t \}}$ resp. $\mathbf{t_{out} = min\{ t_{out} , t \}}$

- early exit if $\mathbf{t_{in}} > \mathbf{t_{out}}$



Implicit surface

ray:

$$P(t) = P_0 + t \cdot \vec{p_1}$$

implicit surface:

$$F(x,y,z) = 0$$

example:

$$(c - \cos ax) \cos z + (y + a \sin ax) \sin z + \\ + \cos a(x+z) = 0$$

- substitution **P(t)** into **F**: $F^*(t) = 0$
- finding roots of the function **F*(t)**
  – sometimes only the **smallest positive root** is needed (the 1st intersection), for **CSG** we need **all roots**

Algebraic surface

ray:

$$P(t) = P_0 + t \cdot \overrightarrow{p_1}$$

algebraic surface of degree **d**:

$$A(x,y,z) = \sum_{i,j,k=0}^{i+j+k \le d} a_{ijk} \cdot x^i y^j z^k = 0$$

example **(toroid with radii a, b):**

$$T_{ab}(x,y,z) = \left(x^2 + y^2 + z^2 - a^2 - b^2\right)^2 - 4a^2\left(b^2 - z^2\right)$$

- after substitution **P(t)** into **A**: **A*(t) = 0**
- **A*** is a polynomial of degree **d** (at most)

Quadric (d=2)

general quadric:

$$x^T Q x = 0 \qquad x = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

after substitution of P(t):

$$a_2 t^2 + a_1 t + a_0 = 0,$$

where $a_2 = P_1^T Q P_1, \quad a_1 = 2 P_1^T Q P_0, \quad a_0 = P_0^T Q P_0$

Quadric of revolution

quadric of revolution in standard position:

$$x^2 + y^2 + az^2 + bz + c = 0$$

sphere:

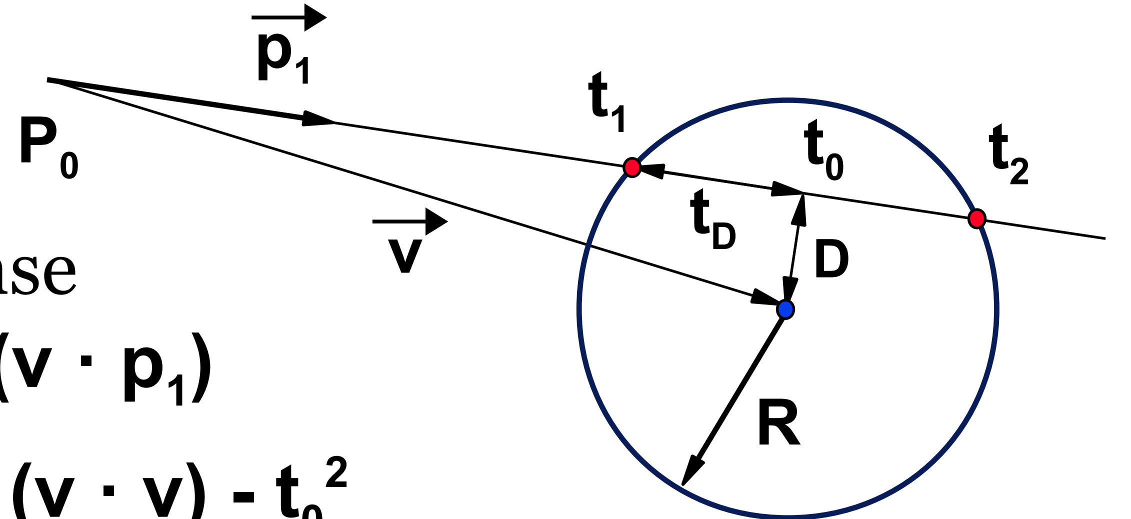
$$x^2 + y^2 + z^2 - 1 = 0,$$

after substitution of P(t):

$$t^2 (P_1 \cdot P_1) + 2t (P_0 \cdot P_1) + (P_0 \cdot P_0) - 1 = 0$$

Sphere (geometric solution)

$$P(t) = P_0 + t \cdot \vec{p_1}$$



- center of the subtense

$$t_0 = (v \cdot p_1)$$

- distance $D^2 = (v \cdot v) - t_0^2$
- inclination $t_D^2 = R^2 - D^2$
- for $t_D^2 = 0$ there is one tangent point $P(t_0)$
- for $t_D^2 > 0$ two intersections exist: $P(t_0 \pm t_D)$
- negative case: **9±**, **6***, **1<**, positive addit.: **2±**, **1 sqrt**

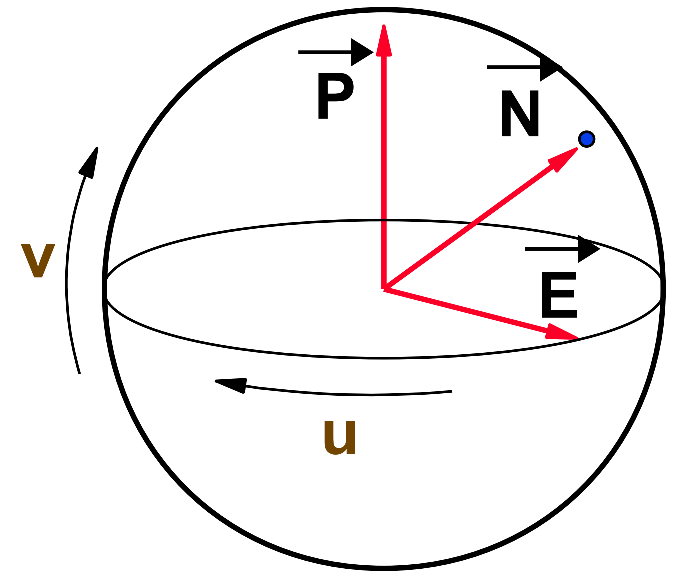
Inverse transformation on the sphere

sphere:

$(x-x_C)^2+(y-y_C)^2+(z-z_C)^2 = R^2$

pole dir: $\vec{P}$, *equator dir:* $\vec{E}$

$(P \cdot E) = 0$



input: **N, P, E**

result: **[u,v] from [0,1]²**

$$\Phi = \arccos\left(-N \cdot P\right), \quad \theta = \frac{\arccos\left[\left(N \cdot E\right) / \sin\Phi\right]}{2\pi}$$

$$v = \Phi / \pi, \quad \left(P \times E\right) \cdot N > 0 \Rightarrow u = \theta, \quad \text{else} \quad u = 1 - \theta$$

Cylinder and cone

unit cylinder and unit cone in basic position:

$$x^2 + y^2 - 1 = 0 \qquad x^2 + y^2 - z^2 = 0$$

after substitution **P(t)** *for the cylinder:*

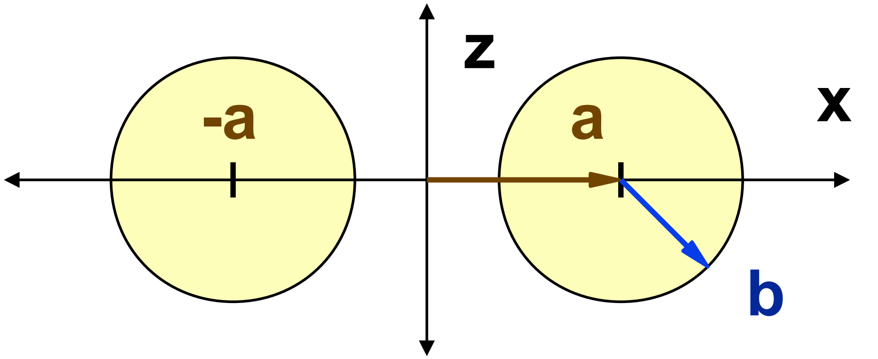
$$t^2\left(x_1^2 + y_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1\right) + x_0^2 + y_0^2 - 1 = 0$$

after substitution **P(t)** *for the cone:*

$$t^2\left(x_1^2 + y_1^2 - z_1^2\right) + 2t\left(x_0 x_1 + y_0 y_1 - z_0 z_1\right) +$$

$$+ x_0^2 + y_0^2 - z_0^2 = 0$$

Toroid



Two circles in the **xz** *plane*:

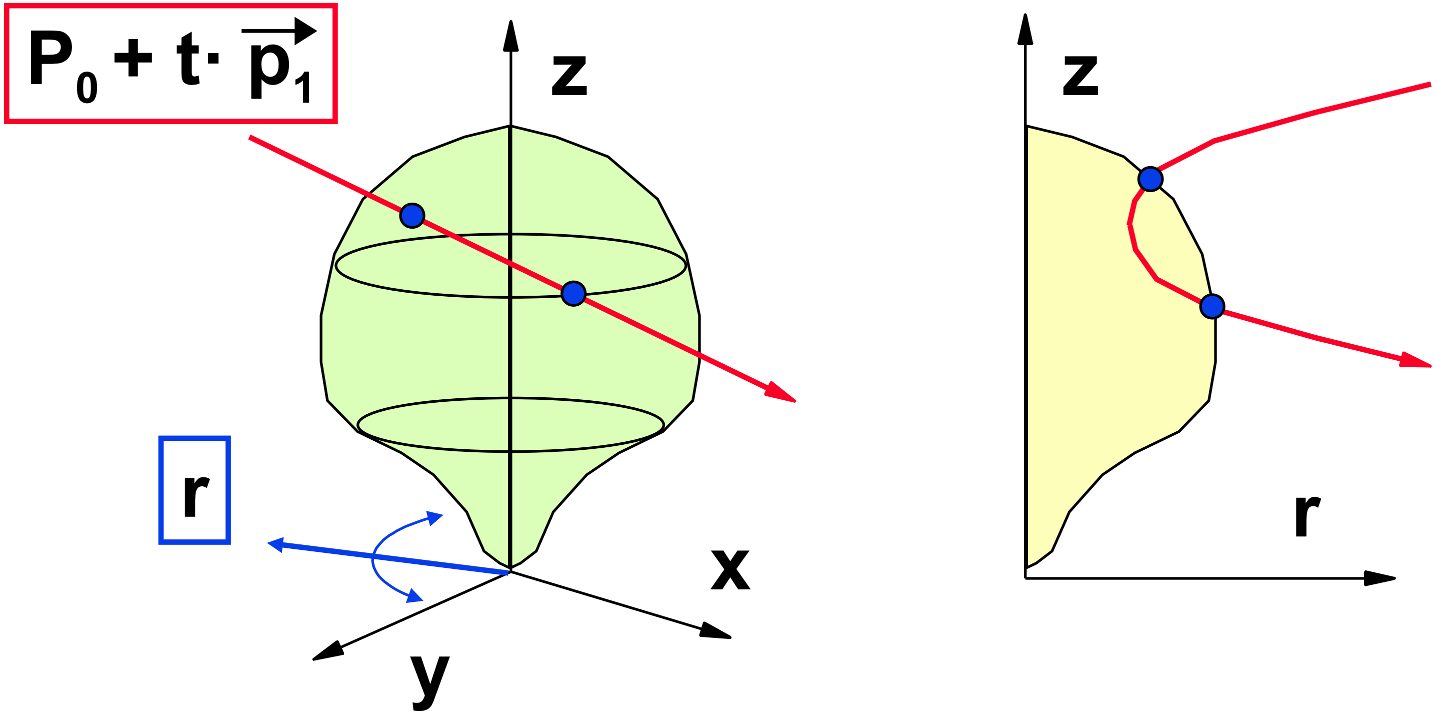
$$\left[(x-a)^2+z^2-b^2\right]\cdot\left[(x+a)^2+z^2-b^2\right]=0$$

$$\left[x^2+z^2-\left(a^2+b^2\right)\right]^2=4a^2\left(b^2-z^2\right)$$

After substitution $r^2=x^2+y^2$ *for* x^2 *– the 4*th *degree equation:*

$$\left(x^2+y^2+z^2-a^2-b^2\right)^2-4a^2\left(b^2-z^2\right)=0$$

Surface of revolution



equation of the ray in **rz** *plane*:

$$r^2 = x^2 + y^2 = (x_0 + x_1 t)^2 + (y_0 + y_1 t)^2$$

$$z = z_0 + z_1 t$$

Ray in rz plane

After elimination of **t**: $\mathbf{ar^2 + bz^2 + cz + d = 0}$ **(1)**

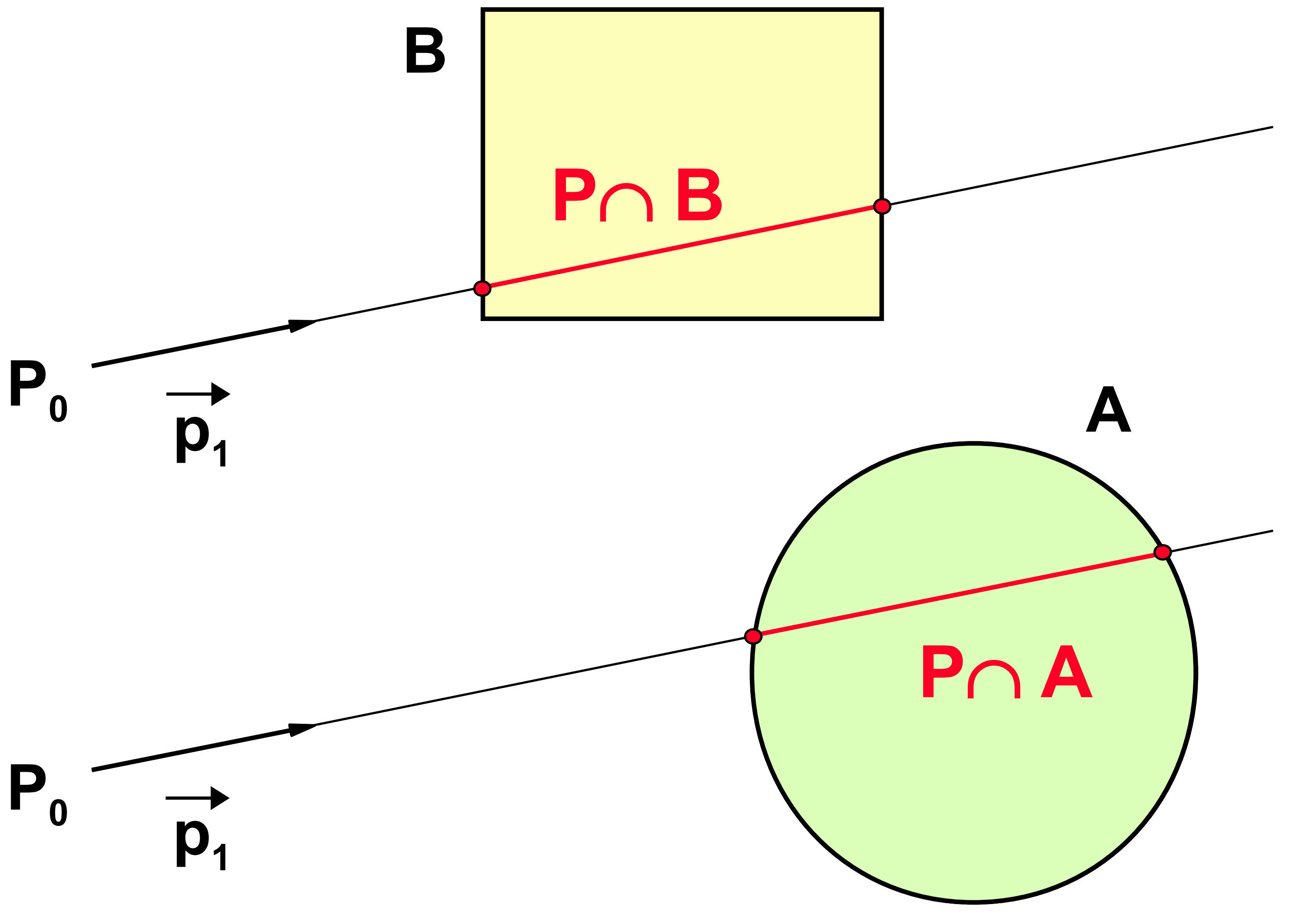
$$a = -z_1^2$$
$$b = x_1^2 + y_1^2$$
$$c = 2(z_1 e - z_0 b)$$
$$d = z_0(z_0 b - 2 z_1 e) + f z_1^2$$
$$e = x_0 x_1 + y_0 y_1$$
$$f = x_0^2 + y_0^2$$

- after substitution of parametric curve **K(s)** into **(1)** we get an equation **K*(s) = 0**
- **K*** has got a double degree (compared to **K**)

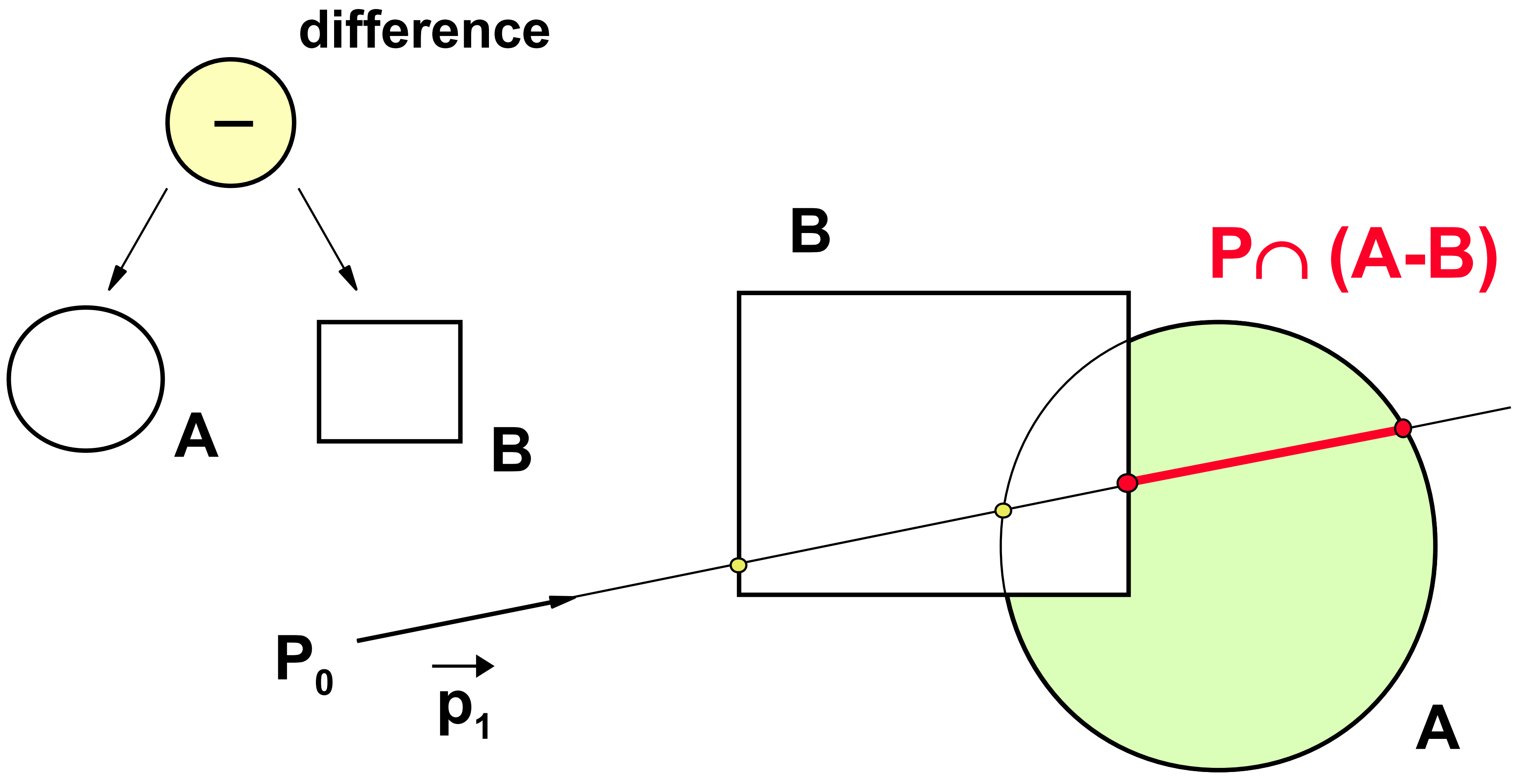
CSG representation

- **primitive solids** are easy
  - convex objects – only two intersections
- **set operations** are performed in the the **1D ray-space**:
  - distributivity: $\mathbf{P \cap (A\text{-}B) = (P \cap A) - (P \cap B)}$
  - general ray-scene intersection is a collection of line segments (intervals in 1D ray-space)
- **geometric transformations**:
  - inverse transformation applied to a ray

Intersections P∩A, P∩B



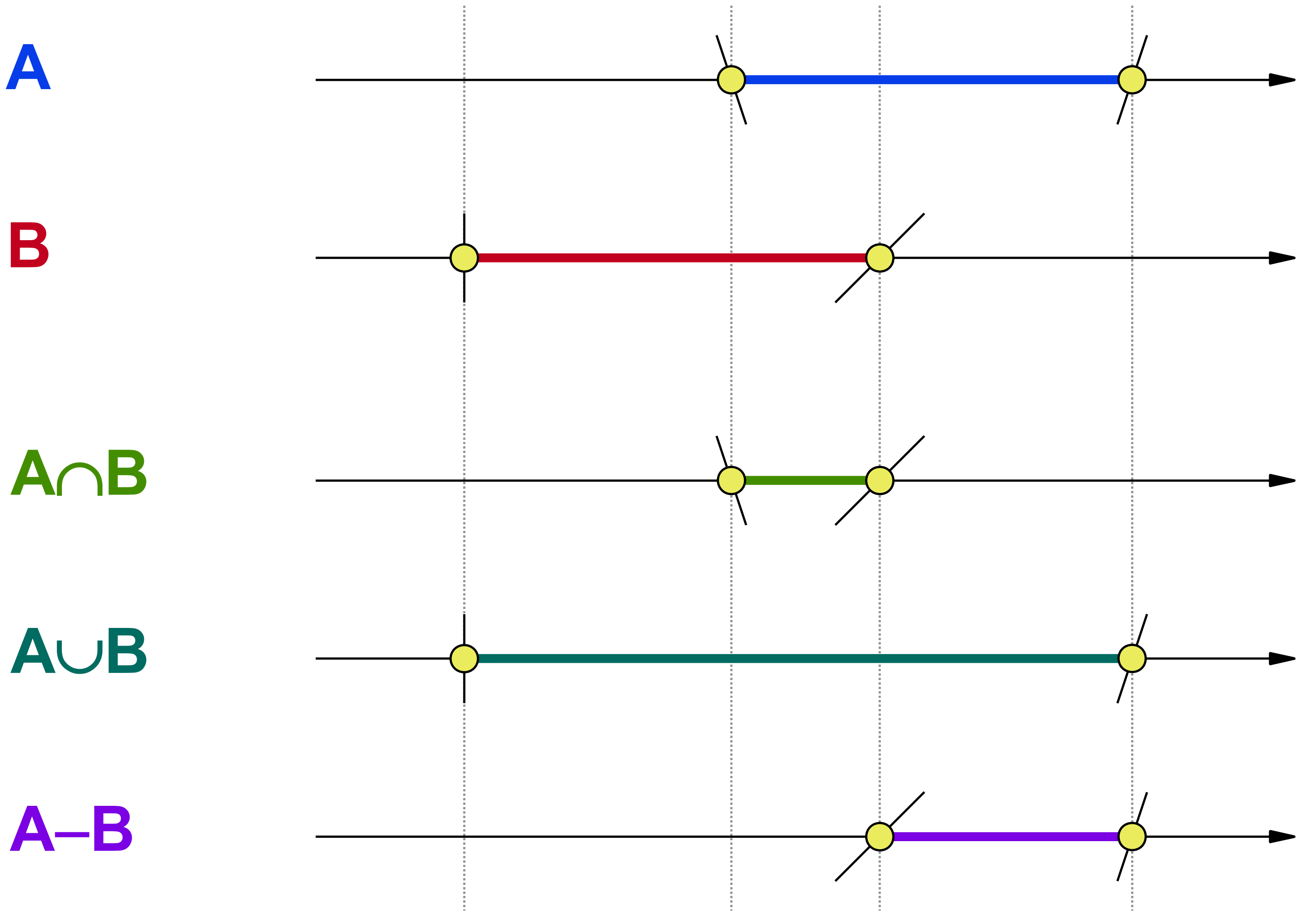
Intersection P∩(A−B)



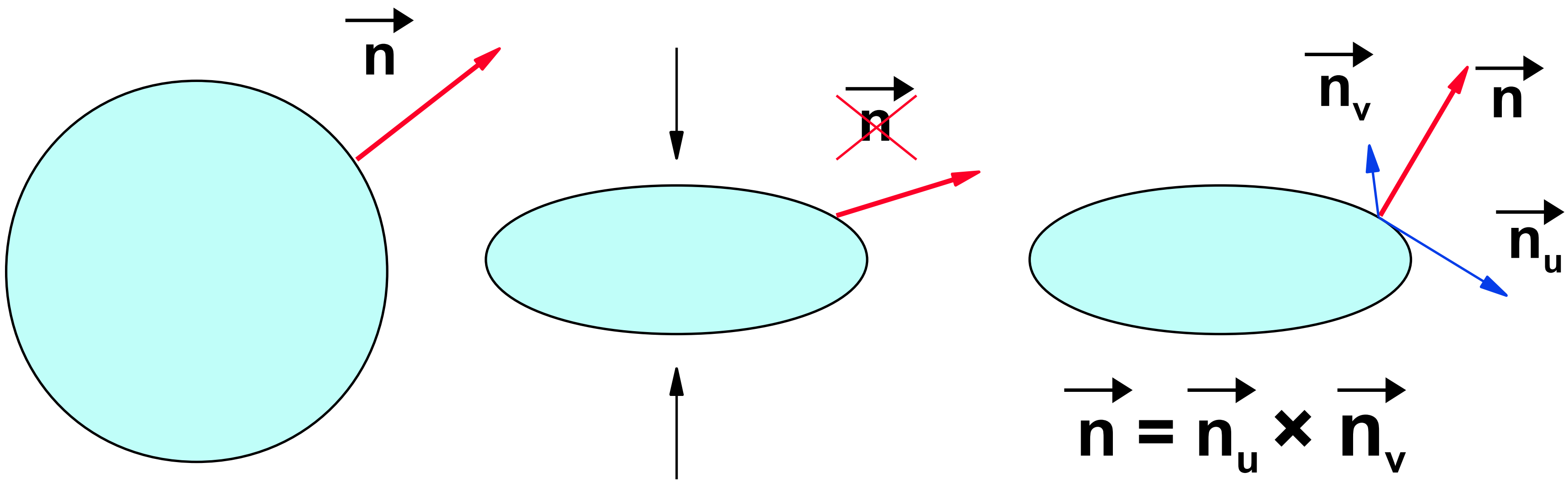
Implementation

- **ray**:
  - origin $\mathbf{P_0}$ and direction $\overrightarrow{\mathbf{p_1}}$
  - transforms with inverse matrices $\mathbf{T_i^{-1}}$ (could not be efficient enough … 1 transformation: **15+**, **18***)
- **ray vs. scene intersection** (partial & final):
  - ordered list of **t** parameter in ray-space: [$\mathbf{t_1}$, $\mathbf{t_2}$, $\mathbf{t_3}$, ..]
- **set operation**:
  - generalized merging of ordered lists **[** $\mathbf{t_i}$ **]**
- **transformation of normal vectors!**

Set operations on the ray



Normal vector transformation



$\vec{n} = \vec{n}_u \times \vec{n}_v$

- general **afine transformation doesn't keep angles**
  - **two tangent vectors** instead a normal
  - **tangent vectors** transformed by **3×3** submatrix only!
- alternative matrix for **normal vectors**: $M_n = (M^{-1})^T$

References

- **A. Glassner**: ***An Introduction to Ray Tracing***, Academic Press, London 1989, 35-119
- **J. Foley, A. van Dam, S. Feiner, J. Hughes**: ***Computer Graphics, Principles and Practice***, 712-714