Adjoint-Driven Russian Roulette and Splitting in Light Transport Simulation Supplemental Document - Draft

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1 Introduction

One purpose of this supplemental document[1] is to introduce the zero-variance transport rules [Křivánek and d'Eon 2014; Hoogenboom 2008](Sec. (3)) and to show that following these rules results in an important principle (Eq. (5) in the paper) that governs the particle weights in the ZV simulation (Sec. 4). Note that our ADRRS is based on this principle: we design the termination/splitting rate q so that we keep this principle even if we do not use the ZV emitting and scattering probabilities.

The other purpose is the derivation of the Eq. (12) from the paper which allows us to explain the importance sampling properties of our ADRRS and to study its relation to the ZV scheme (Sec. 5).

In the following section, we review formal definitions of the basic light transport concepts such as rendering and visual importance transport equations and their related measurements.

2 Light Transport Background

2.1 Rendering Equation and Light Tracing

Light transport in a scene without participating media is described by the *rendering equation* [Kajiya 1986; Dutré et al. 2006]:

$$L_\mathrm{o}(\mathbf{y},\omega_\mathrm{o}) = L^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega_\mathrm{o}) + \underbrace{\int_\Omega L_\mathrm{i}(\mathbf{y},\omega_\mathrm{i}) f_\mathrm{s}(\mathbf{y},\omega_\mathrm{i}\!\to\!\omega_\mathrm{o})|\cos\theta_\mathrm{i}|\,\mathrm{d}\omega_\mathrm{i}}_{L^\mathrm{r}_\mathrm{o}(\mathbf{y},\omega_\mathrm{o})}\,.$$

Here $L_\mathrm{o}(\mathbf{y},\omega_\mathrm{o})$ and $L^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega_\mathrm{o})$ are, respectively, the total and the self-emitted outgoing radiance from a surface point $\mathbf{y}$ in a direction ω_o, f_s denotes the bidirectional scattering distribution function (BSDF), θ_i is the angle between the surface normal at $\mathbf{y}$ and an incident direction ω_i, and Ω is the unit sphere. We use the arrow notation in f_s to mark the direction of light flow. The incident radiance $L_\mathrm{i}(\mathbf{y},\omega_\mathrm{i})$ at the point $\mathbf{y}$ visible from a point $\mathbf{x}$ in the direction ω_i is equal to the outgoing radiance $L_\mathrm{o}(\mathbf{x},-\omega_\mathrm{i})$. L^r_o denotes the part of the outgoing radiance that is only due to surface reflection at $\mathbf{y}$.

The *light tracing* algorithm [Dutré and Willems 1994] estimates the measurement equation (i.e. pixel value) I

$$I = \int_\mathcal{M}\int_\Omega L_\mathrm{i}(\mathbf{y},\omega)\,W^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega)\,|\cos\theta|\,\mathrm{d}\omega\,\mathrm{d}\mathbf{y}, \tag{1}$$

using the following MC estimator [Veach 1997, 4.A]:

$$\langle I\rangle = \frac{1}{N}\sum_k \nu_\mathrm{i}(\mathbf{y}_k,\omega_k)\,W^\mathrm{e}_\mathrm{o}(\mathbf{y}_k,\omega_k). \tag{2}$$

Here, $W^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega)$ is the self-emitted visual importance. Evaluating this estimator involves following random paths of N particles emitted from the light sources. The above sum is updated when a particle k with its *weight* $\nu_\mathrm{i}(\mathbf{y}_k,\omega_k)$, coming from a direction ω_k, collides at a location $\mathbf{y}_k$. Note that the above estimator corresponds to light tracing without explicit connections to the camera (next event estimation).

2.2 Visual Importance Transport Equation and Path Tracing

The visual importance transport is governed by the following transport equation

$$W_\mathrm{o}(\mathbf{y},\omega_\mathrm{o}) = W^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega_\mathrm{o}) + \underbrace{\int_\Omega W_\mathrm{i}(\mathbf{y},\omega_\mathrm{i}) f_\mathrm{s}(\mathbf{y},\omega_\mathrm{o}\!\to\!\omega_\mathrm{i})|\cos\theta_\mathrm{i}|\,\mathrm{d}\omega_\mathrm{i}}_{W^\mathrm{r}_\mathrm{o}(\mathbf{y},\omega_\mathrm{o})}, \tag{3}$$

which is *adjoint* to the rendering equation [Christensen 2003]. Here, W_i is incident visual importance. Recall, that we use a convention, depicted in Fig. 2 in the paper, that ω_o always points in the direction of the transported quantity and that the arrow in f_s depicts the direction of light transport.

Path tracing is perhaps the most popular light transport simulation method used in computer graphics, which can be though of as following particles of visual importance starting from the camera. By using the MC estimator given by Eq. (1) in the paper, the algorithm estimates the pixel value I given by the visual importance *measurement equation* [Veach 1997]

$$I = \int_\mathcal{M}\int_\Omega W_\mathrm{i}(\mathbf{y},\omega)\,L^\mathrm{e}_\mathrm{o}(\mathbf{y},\omega)\,|\cos\theta|\,\mathrm{d}\omega\,\mathrm{d}\mathbf{y}, \tag{4}$$

where $\mathcal{M}$ is the scene surface.

3 Zero-Variance Random Walk Rules

To make the radiance measurement equation (Eq. (2)) a zero-variance estimator, we must obey the following rules:

- Emit light particles according to the pdf proportional to the product of incident importance distribution at the light sources and cosine-weighted outgoing self-emitted radiance:

$$p^\mathrm{e}_\mathrm{zv}(\mathbf{x},\omega_\mathrm{o}) = \frac{W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o})\,L^\mathrm{e}_\mathrm{o}(\mathbf{x},\omega_\mathrm{o})\,|\cos\theta_\mathrm{o}|}{I}, \tag{5}$$

  where the division by I makes the pdf integrate to 1 (see Eq. (4), the visual importance measurement equation).

- At each scattering event, sample the new direction from the pdf proportional to the product of the cosine-weighted BSDF f_s and the incident visual importance distribution:

$$p_\mathrm{zv}(\omega_\mathrm{o}|\mathbf{x}) = \frac{W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o}) f_\mathrm{s}(\mathbf{x},\omega_\mathrm{i}\!\to\!\omega_\mathrm{o})|\cos\theta_\mathrm{o}|}{W^\mathrm{r}_\mathrm{o}(\mathbf{x},\omega_\mathrm{i})}. \tag{6}$$

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[1]This is a draft. The final version can be found here: http://cgg.mff.cuni.cz/~jirka/papers/2016/adrrs/index.htm

The division by reflected importance W_o^r at $\mathbf{x}$ makes the pdf integrate to 1 as can be seen from Eq. (3) by replacing the role of ω_i and ω_o (we trace from light sources here while Eq. (3) is defined with respect to tracing from the camera).

- Use the following survival probability

$$q_{\mathrm{zv}}(\mathbf{y},\omega_\mathrm{i}) = 1 - \frac{W_\mathrm{o}^\mathrm{e}(\mathbf{y},\omega_\mathrm{i})}{W_\mathrm{o}(\mathbf{y},\omega_\mathrm{i})} = \frac{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})}{W_\mathrm{o}(\mathbf{y},\omega_\mathrm{i})}, \tag{7}$$

that allows the walk to terminate only on the camera sensor.

- Contribute to the estimator only upon termination (i.e. with the probability $1 - q_{\mathrm{zv}}$).

4 Particle Weight in Zero-Variance Schemes

In this section, we show that the incident weight $\hat{\nu}(\mathbf{y},\omega_\mathrm{i})$ of a particle that follows a zero-variance random walk and survives the collision at $\mathbf{y}$ is equal to (Eq. (5) in the paper)

$$\hat{\nu}(\mathbf{y},\omega_\mathrm{i}) = \frac{I}{\Psi_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})}. \tag{8}$$

Here, I is the computed pixel value and adjoint $\Psi_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})$ reflected from a point $\mathbf{y}$ into the direction ω_i depends on the tracing type. If we trace from light sources it stands for visual importance W while if we trace from the camera it stands for radiance L. Note that the adjoint Ψ is sum of source adjoint Ψ^e and reflected adjoint Ψ^r. Without loss of generality we show validity of Eq. (8) on an example zero-variance scheme applied to *light tracing*.

4.1 Particle Weight Invariant

Under the light tracing zero-variance scheme above, the incident weight of a surviving particle at $\mathbf{y}$ reads

$$\boxed{\hat{\nu}(\mathbf{y},\omega_\mathrm{i}) = \frac{I}{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})}.} \tag{9}$$

Note that this is only a light tracing instance of the general case (Eq. (8)). We now show the validity of Eq. (9) by induction.

Proof of Eq. (9)**.** We first show that this induction hypothesis is valid after emission. For this purpose, we consider that $\mathbf{x}$ is an emission event and precedes the collision at $\mathbf{y}$. The notation corresponds to Fig. 1 except that there is no $\omega_\mathrm{i}^\mathrm{x}$ direction as $\mathbf{x}$ is now the emission event. Plugging emission probability (Eq. (5)) into the initial weight of an emitted particle that is given by

$$\nu_\mathrm{o}^\mathrm{e}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x}) = \frac{L_\mathrm{o}^\mathrm{e}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})|\cos\theta_\mathbf{x}|}{p^\mathrm{e}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})}, \tag{10}$$

leads to the following particle weight after emission:

$$\nu_\mathrm{o}^\mathrm{e}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x}) = \frac{I}{W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})}. \tag{11}$$

Here, $p^\mathrm{e}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})$ is the joint pdf of sampling the initial particle position $\mathbf{x}$ and the direction $\omega_\mathrm{o}^\mathrm{x}$. When the particle survives the collision at $\mathbf{y}$, the incident weight becomes

$$\hat{\nu}(\mathbf{y},\omega_\mathrm{i}) = \frac{\nu_\mathrm{i}(\mathbf{y},\omega_\mathrm{i})}{q_{\mathrm{zv}}(\mathbf{y},\omega_\mathrm{i})} = \frac{\nu_\mathrm{o}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})}{q_{\mathrm{zv}}(\mathbf{y},\omega_\mathrm{i})}. \tag{12}$$

The second equality follows from the fact that $\nu_\mathrm{o}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x}) = \nu_\mathrm{i}(\mathbf{y},\omega_\mathrm{i})$ (as shown in Fig. 1). Finally, by using the ZV surviving probability q_{zv} (Eq. (7)) and the particle weight after emission (Eq. (11)) in Eq. (12), we get

$$\hat{\nu}(\mathbf{y},\omega_\mathrm{i}) = \frac{I}{W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})}\frac{W_\mathrm{o}(\mathbf{y},\omega_\mathrm{i})}{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})} = \frac{I}{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})}. \tag{13}$$

The equality in question (Eq. (9)) holds because $W_\mathrm{o}(\mathbf{y},\omega_\mathrm{i}) = W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})$ (see Fig. 1).

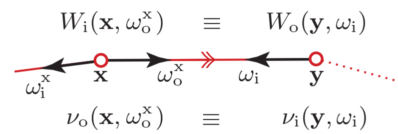

Figure 1: *A light tracing collisions where the collision at* $\mathbf{x}$ *precedes the collision at* $\mathbf{y}$*. The figure depicts important identities between the two events.*

To see that the incident weight of the surviving particle (Eq. (9)) is maintained through a collision at $\mathbf{y}$, we now suppose that $\mathbf{x}$ is a general collision location that precedes $\mathbf{y}$ and that the induction hypothesis holds at $\mathbf{x}$: $\hat{\nu}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x}) = I/W_\mathrm{o}^\mathrm{r}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})$ (see Fig. 1). When we sample an outgoing direction $\omega_\mathrm{o}^\mathrm{x}$ from $\mathbf{x}$ according to ZV scheme, we get the outgoing particle weight

$$\nu_\mathrm{o}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x}) = \hat{\nu}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})\frac{f_\mathrm{s}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x}\!\to\!\omega_\mathrm{o}^\mathrm{x})|\cos\theta_\mathbf{x}|}{p_{\mathrm{zv}}(\omega_\mathrm{o}^\mathrm{x})}.$$

Now we substitute the induction assumption for $\hat{\nu}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})$ and ZV scattering probability (Eq. (6)) for $p_{\mathrm{zv}}(\omega_\mathrm{o}^\mathrm{x})$ to arrive at

$$\nu_\mathrm{o}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x}) = \frac{I}{W_\mathrm{o}^\mathrm{r}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})}\frac{W_\mathrm{o}^\mathrm{r}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})}{W_\mathrm{i}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})}. \tag{14}$$

Similarly as in the case of emission, when we apply the ZV surviving probability $q_{\mathrm{zv}}(\mathbf{y},\omega_\mathrm{i})$ (Eqs. (7)) and this derived outgoing particle weight $\nu_\mathrm{o}(\mathbf{x},\omega_\mathrm{o}^\mathrm{x})$ (Eq. (14)) to express the weight of a particle surviving collision at $\mathbf{y}$ (Eq. (12)), we get Eq. (9) which is what had to be proven.

5 Derivation of Eq. (12) in the paper

We show at an example of light tracing that designing our ADRRS so that

$$\hat{\nu}(\mathbf{y},\omega_\mathrm{i}) = \frac{I}{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})} \tag{15}$$

at each collision $\mathbf{y}$ and an incident direction ω_i results in the following RR/splitting rate q at the collision $\mathbf{y}$:

$$\boxed{q(\mathbf{y},\omega_\mathrm{i}) = \frac{p_{\mathrm{zv}}^\mathrm{r}(\omega_\mathrm{o}^\mathrm{x}|\,\mathbf{x})}{p(\omega_\mathrm{o}^\mathrm{x}|\,\mathbf{x})}.} \tag{16}$$

Note that the version of this equation stated here is more general then Eq. (12) in the paper because it is valid also when $\mathbf{x}$ in on a source or a sensor. This equation says that the termination/splitting rate q at $\mathbf{y}$ is equal to the ratio of a function $p_{\mathrm{zv}}^\mathrm{r}$ defined as

$$p_{\mathrm{zv}}^\mathrm{r}(\omega|\,\mathbf{x}) = \frac{W_\mathrm{o}^\mathrm{r}(\mathbf{y},\omega_\mathrm{i})f_\mathrm{s}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x}\!\to\!\omega)|\cos\theta_\mathbf{x}|}{W_\mathrm{o}^\mathrm{r}(\mathbf{x},\omega_\mathrm{i}^\mathrm{x})}$$

and the actual direction sampling distribution $p(\omega|\,\mathbf{x})$ at $\mathbf{x}$. The only difference between $p_{\mathrm{zv}}^\mathrm{r}(\omega|\,\mathbf{x})$ and the zero-variance scattering distribution $p_{\mathrm{zv}}(\omega|\,\mathbf{x})$ (Eq. (6)) is that the function $p_{\mathrm{zv}}^\mathrm{r}$ does not include direct importance emitted from $\mathbf{y}$. Thus $p_{\mathrm{zv}}(\omega|\,\mathbf{x}) = p_{\mathrm{zv}}^\mathrm{r}(\omega|\,\mathbf{x})$ if $\mathbf{y}$ is not on the camera (or a light source in case of path tracing). Note that the two consecutive collisions $\mathbf{x}$ and $\mathbf{y}$ are illustrated in Fig. 1.

We start our derivation of Eq. (16) by recalling the weight update formula at $\mathbf{y}$ after application of ADRRS (Eq. (3) in the paper):

$$\hat{\nu}(\mathbf{y}, \omega_\mathrm{i}) = \frac{\nu_\mathrm{i}(\mathbf{y}, \omega_\mathrm{i})}{q(\mathbf{y}, \omega_\mathrm{i})}.$$

From this equation and from Eq. (15) we derive the formula for RR/splitting rate q at $\mathbf{y}$ (Eq. (4) in the paper) which, using the identity between ν_i and ν_o (see Fig. 1), can be written as:

$$q(\mathbf{y}, \omega_\mathrm{i}) = \frac{\nu_\mathrm{o}(\mathbf{x}, \omega_\mathrm{o}^\mathrm{x}) W_\mathrm{o}^\mathrm{r}(\mathbf{y}, \omega_\mathrm{i})}{I}. \tag{17}$$

Now we need to express the outgoing particle weight ν_o at $\mathbf{x}$ in terms of path weight $\hat{\nu}$ after application of ADRRS at $\mathbf{x}$. When the particle scatters at $\mathbf{x}$ into a direction $\omega_\mathrm{o}^\mathrm{x}$ its outgoing weight ν_o is computed as follows (Eq. (2)):

$$\nu_\mathrm{o}(\mathbf{x}, \omega_\mathrm{o}^\mathrm{x}) = \hat{\nu}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x}) \frac{f_\mathrm{s}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x} \!\rightarrow\! \omega_\mathrm{o}^\mathrm{x}) |\cos\theta_\mathbf{x}|}{p(\omega_\mathrm{o}^\mathrm{x} | \, \mathbf{x})}. \tag{18}$$

At this point, we write the basic constraint on the particle weight (Eq. (15)) which our ADRRS ensures through termination/splitting at every collision in terms of scattering at $\mathbf{x}$: $\hat{\nu}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x}) = \frac{I}{W_\mathrm{o}^\mathrm{r}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x})}$. When we insert this constraint and Eq. (18) into Eq. (17), we arrive at the following splitting rate q:

$$q(\mathbf{y}, \omega_\mathrm{i}) = \underbrace{\frac{W_\mathrm{o}^\mathrm{r}(\mathbf{y}, \omega_\mathrm{i}) f_\mathrm{s}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x} \!\rightarrow\! \omega_\mathrm{o}^\mathrm{x}) |\cos\theta_\mathbf{x}|}{W_\mathrm{o}^\mathrm{r}(\mathbf{x}, \omega_\mathrm{i}^\mathrm{x})}}_{p_\mathrm{zv}^\mathrm{r}(\omega_\mathrm{o}^\mathrm{x} | \, \mathbf{x})} \frac{1}{p(\omega_\mathrm{o}^\mathrm{x} | \, \mathbf{x})} \tag{19}$$

This equality what had to been proven follows from the identity between W_i and W_o (which similarly holds for W_i^r and W_o^r) depicted in Fig. 1.

5.1 Zero-variance sampling and ADRRS

If we use ADRRS on top of the ZV scheme, i.e. $p(\omega | \, \mathbf{x}) = p_\mathrm{zv}(\omega | \, \mathbf{x})$ (see Eq. (6)) for all ω and we also use the ZV emission rule (Eq. (5)), then Eq. (19) becomes exactly the ZV surviving probability (Eq. (7)). Even though this suggests that ADRRS does not violate the ZV scheme when the ZV scattering and emission rules are obeyed, there is one subtle breach of the ZV rules. While the ZV scheme does not allow contribution from particles that reached a light source (or sensor) but were not terminated, our ADRRS approach always counts the contribution from such particles. Note that this introduced variance has a very limited impact in practice as it only matters on light sources (or sensors).

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