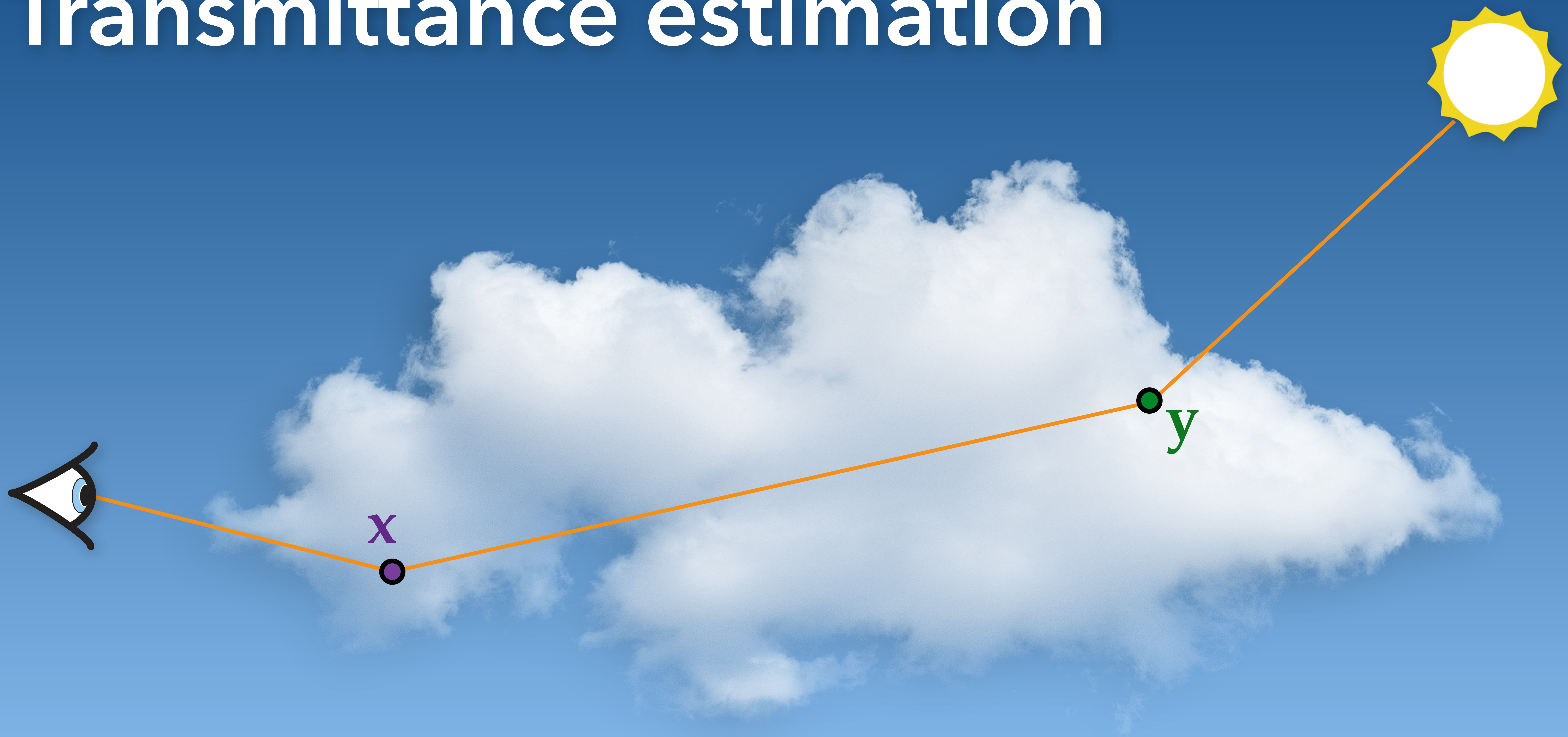
Monte Carlo Methods for Physically Based Volume Rendering

Transmittance estimation

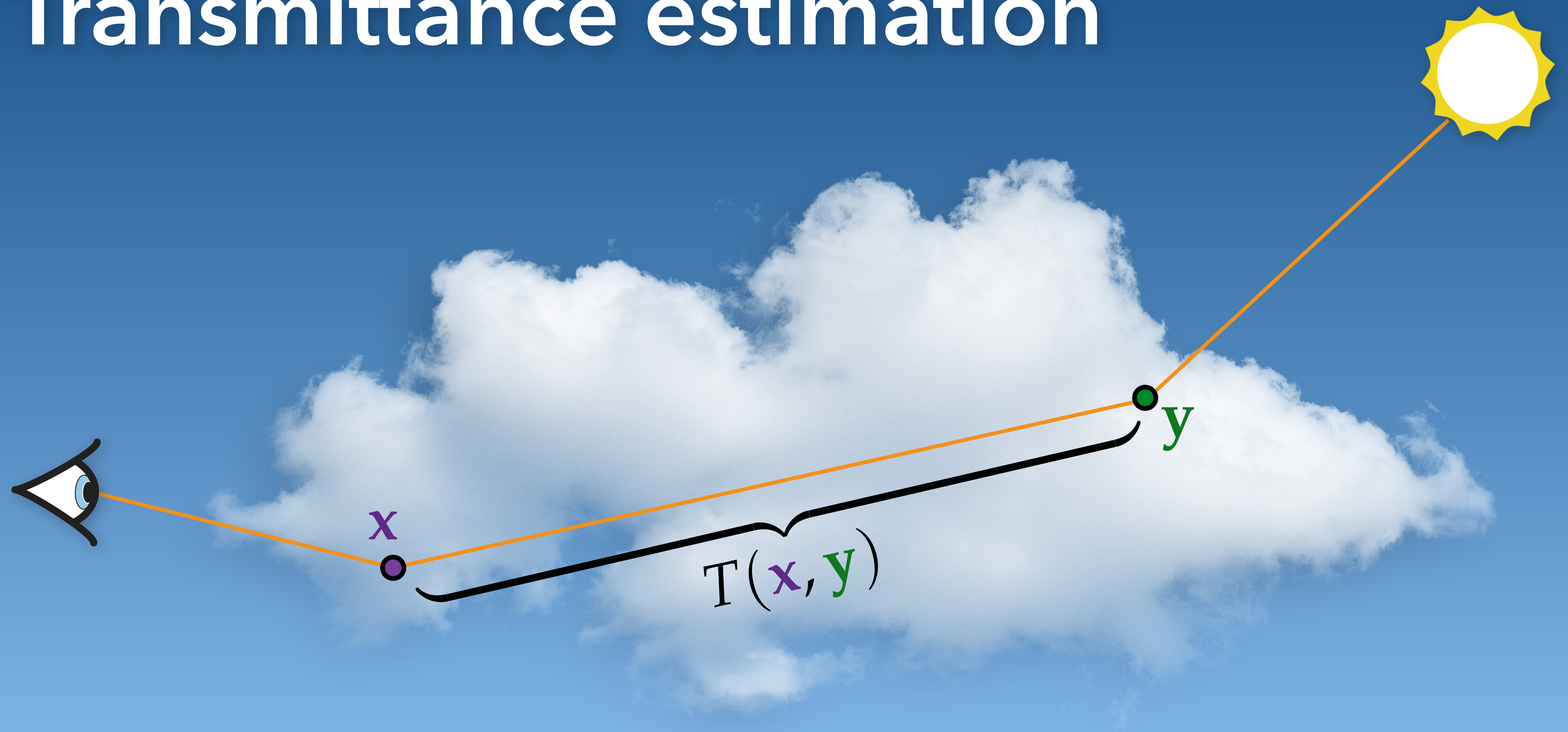
Wojciech Jarosz
wjarosz@dartmouth.edu

DARTMOUTH
VISUAL COMPUTING LAB

Transmittance estimation



Transmittance estimation



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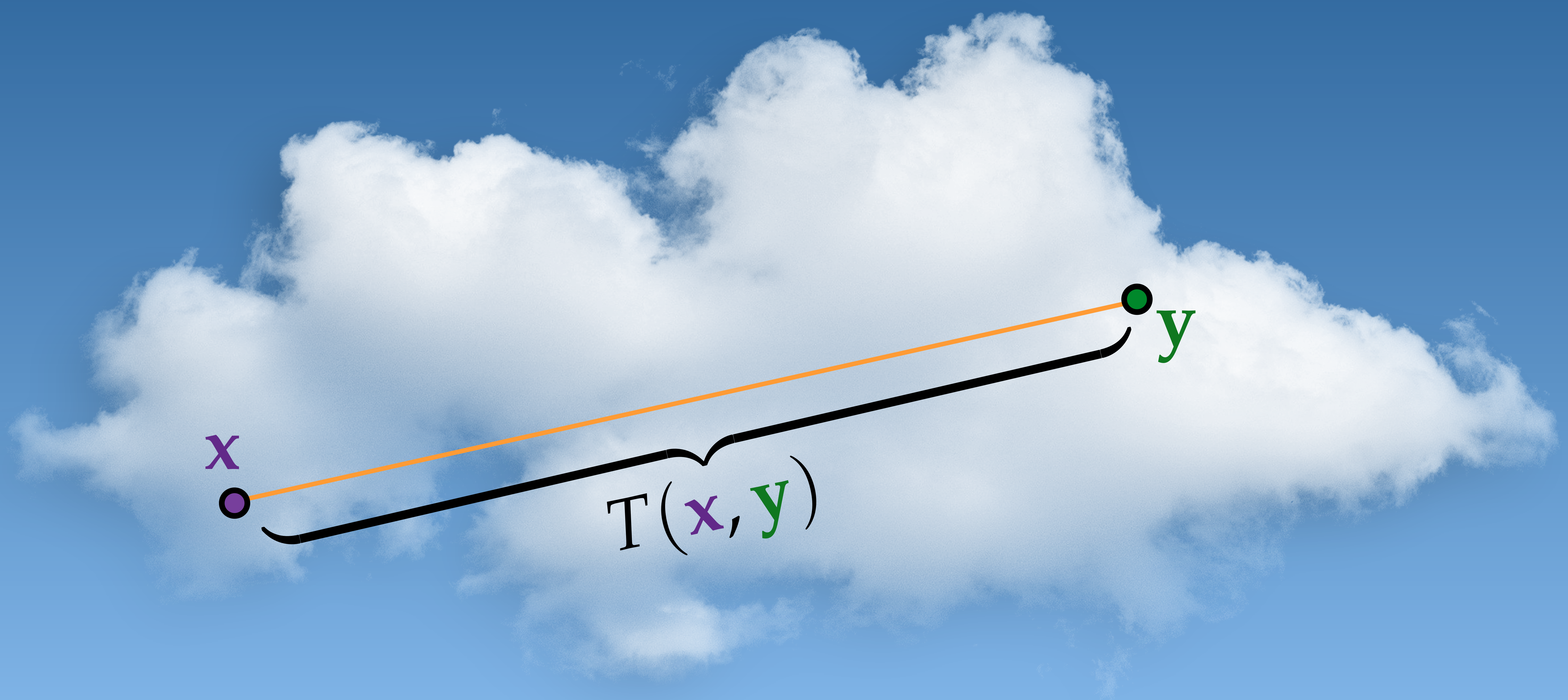

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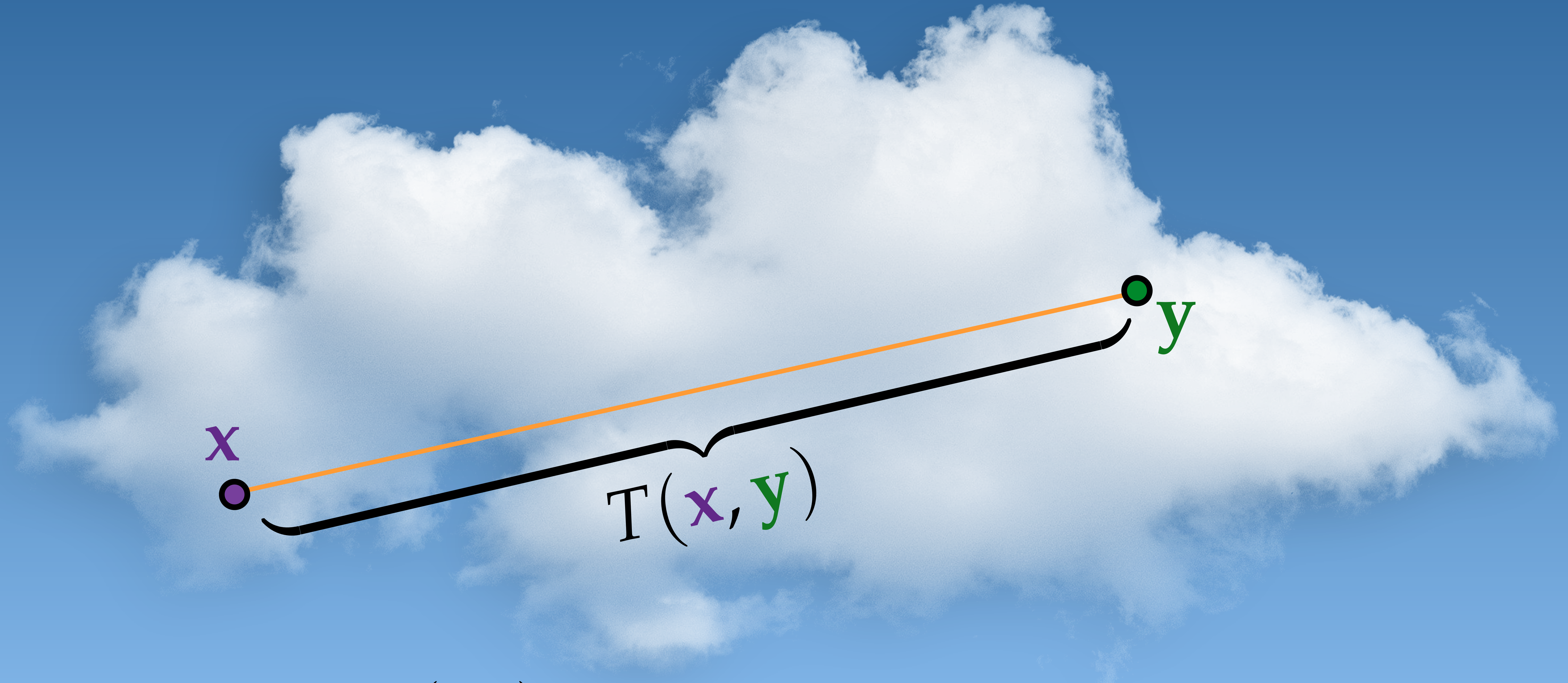
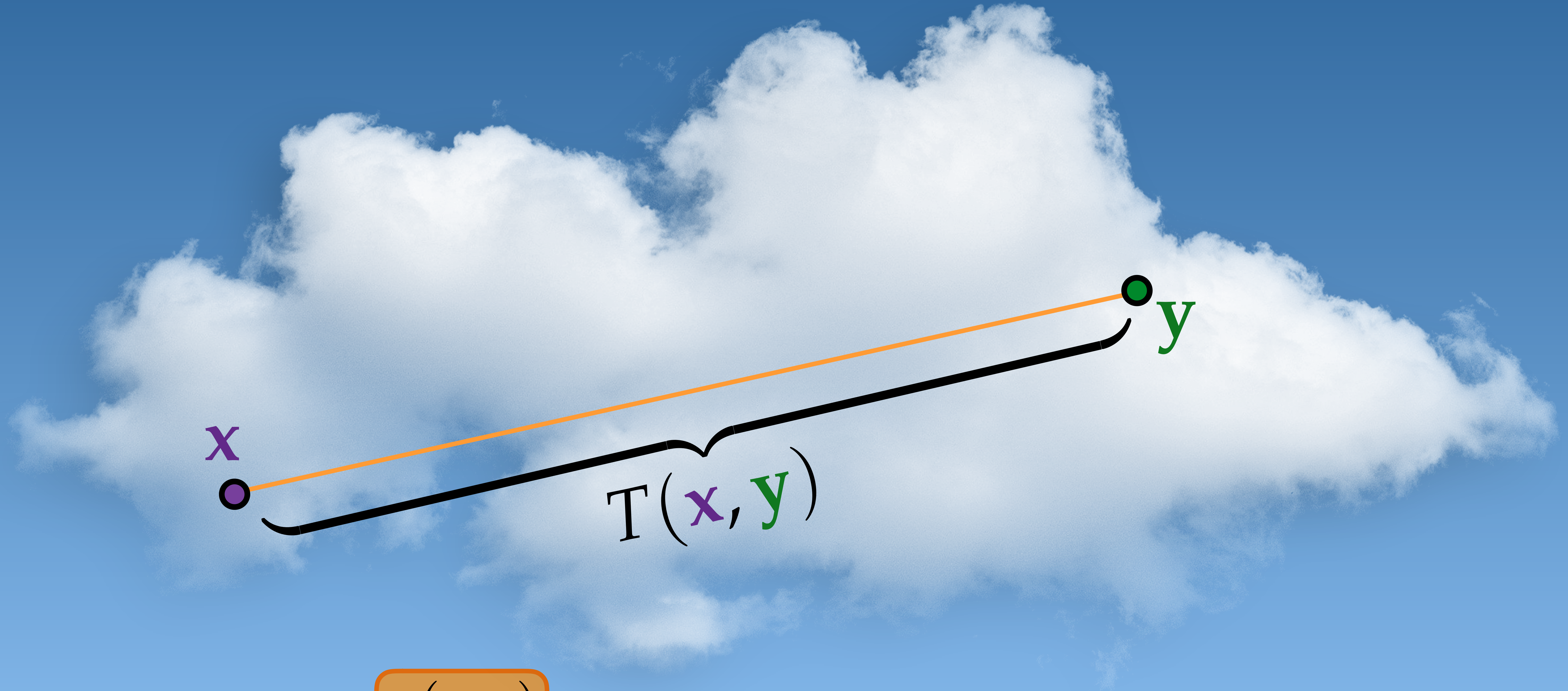


$$T(\mathbf{x}, \mathbf{y}) = e^{-\tau(\mathbf{x}, \mathbf{y})}$$

transmittance

Transmittance estimation



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transmittance

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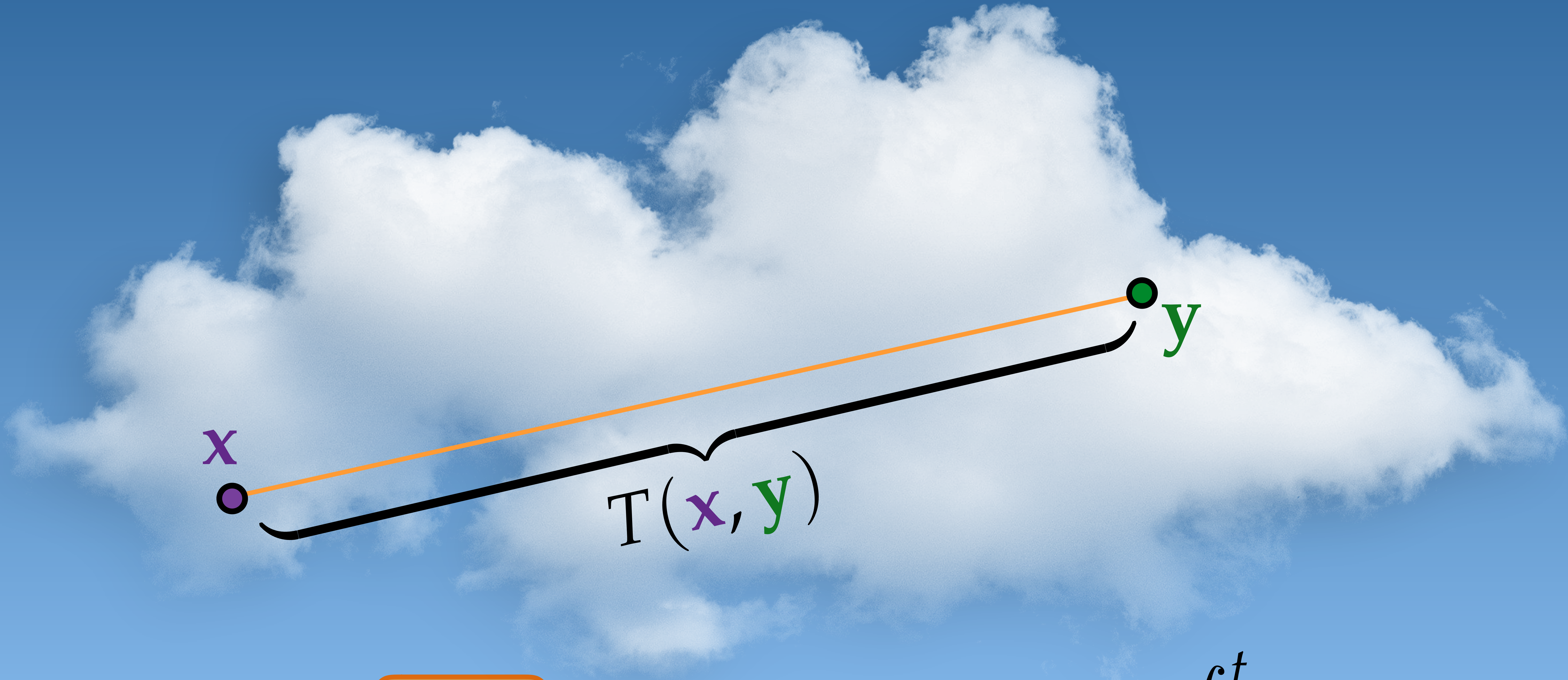


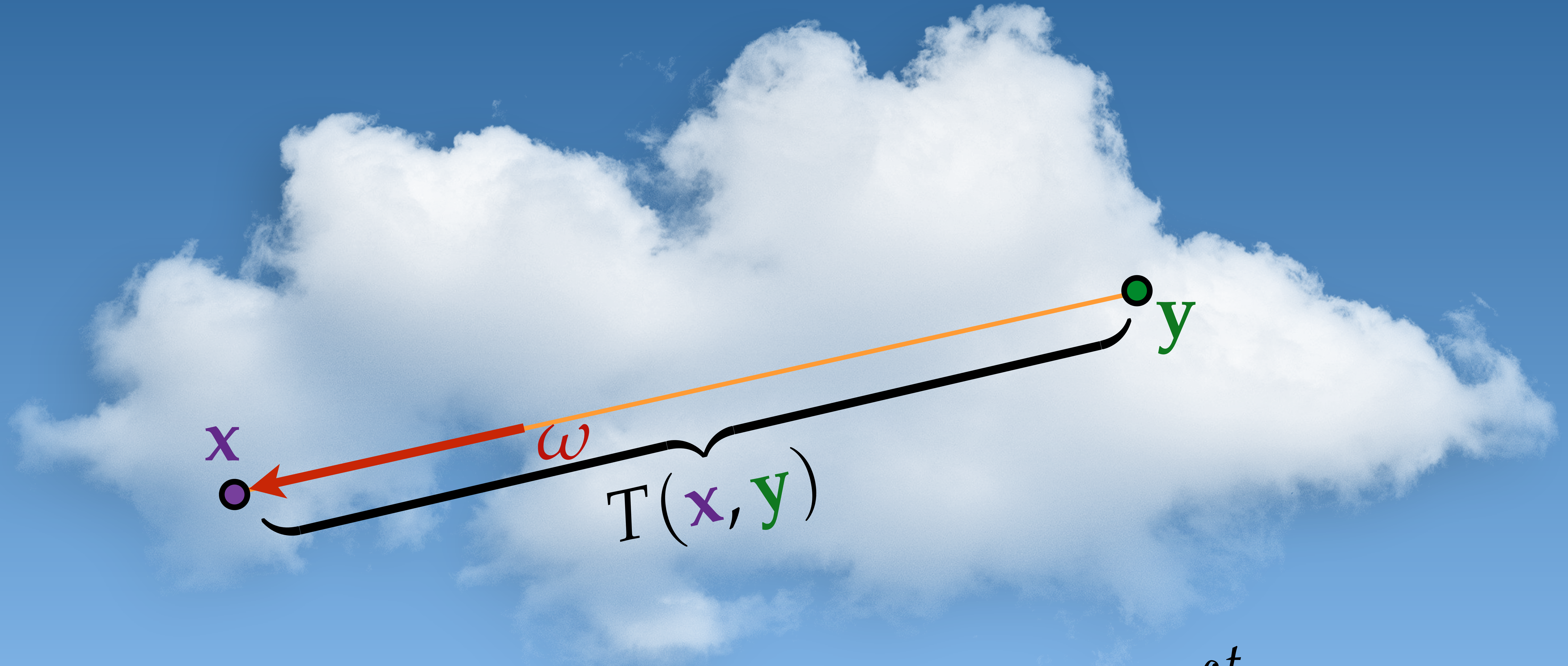
$$T(\mathbf{x}, \mathbf{y}) = e^{-\tau(\mathbf{x}, \mathbf{y})}$$

transmittance

$$\tau(\mathbf{x}, \mathbf{y}) = \int_0^t \mu_t(\mathbf{x} - s\omega)\, ds$$

"optical thickness"

Transmittance estimation



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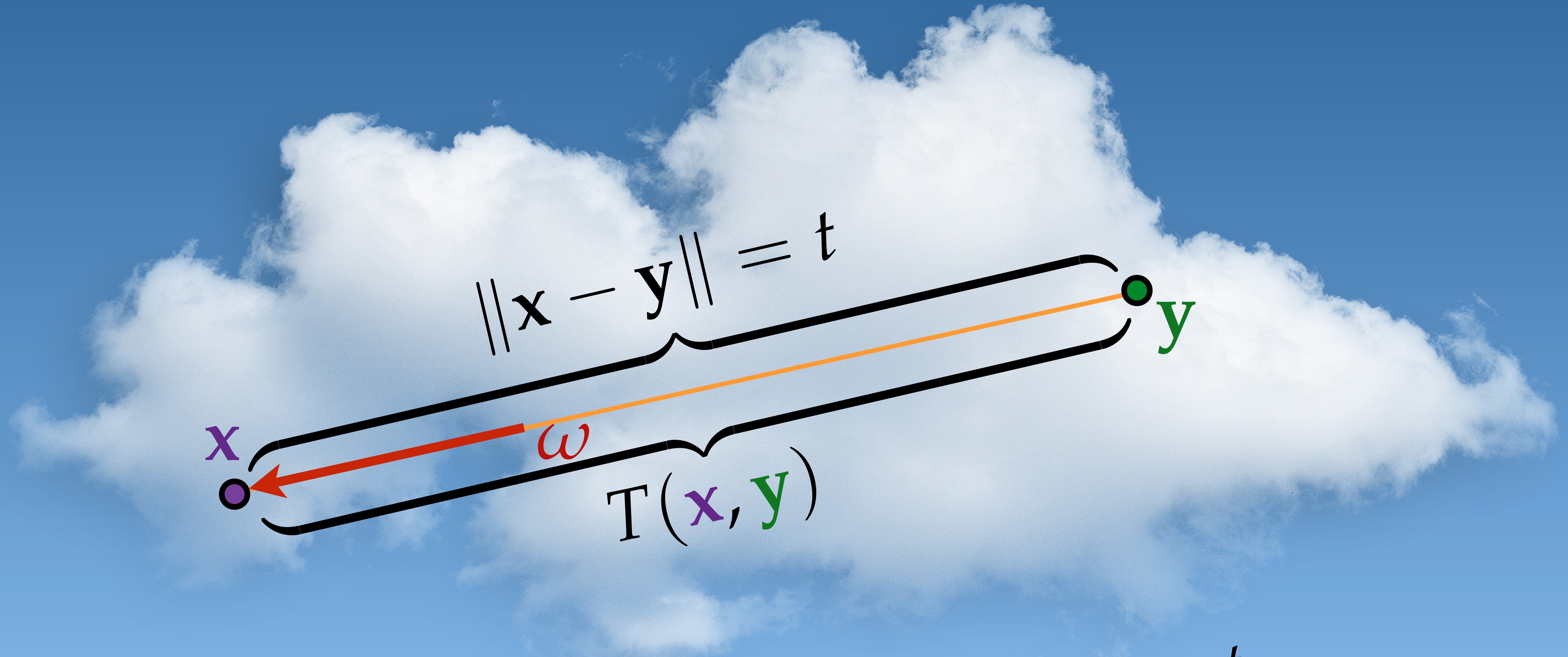

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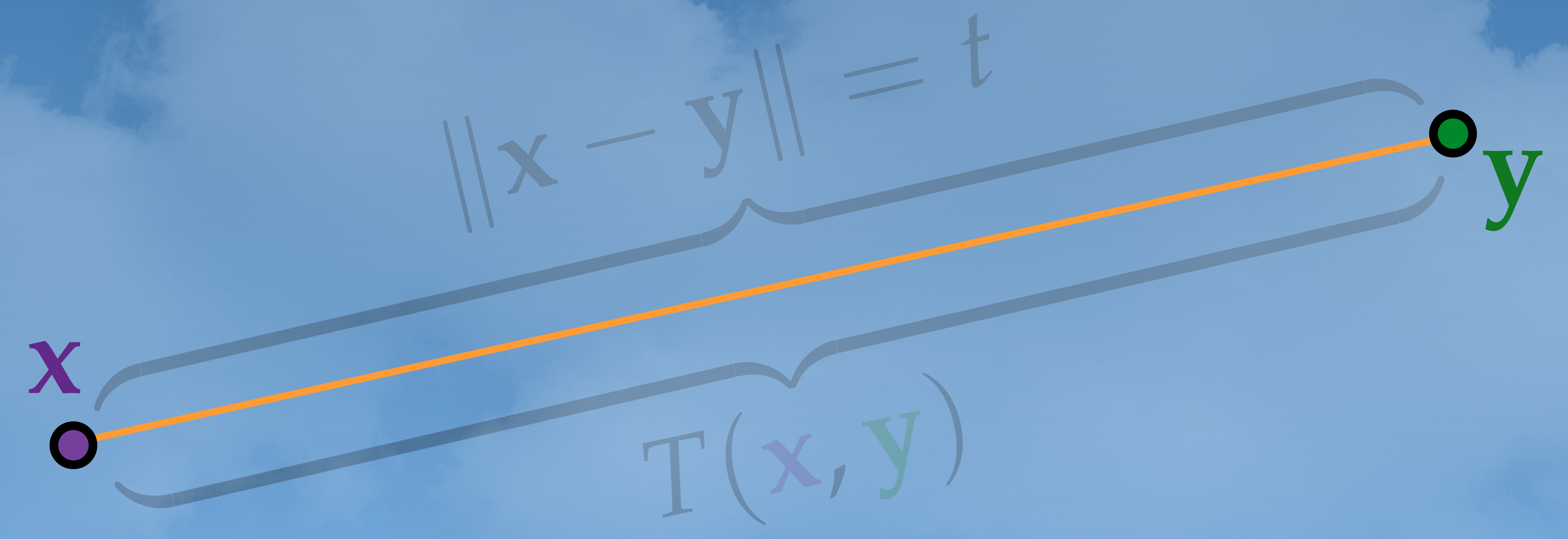
"optical thickness"

Transmittance estimation

1. Estimators integrating **optical thickness**



$$T(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{-\tau(\mathbf{x}, \mathbf{y})}$$

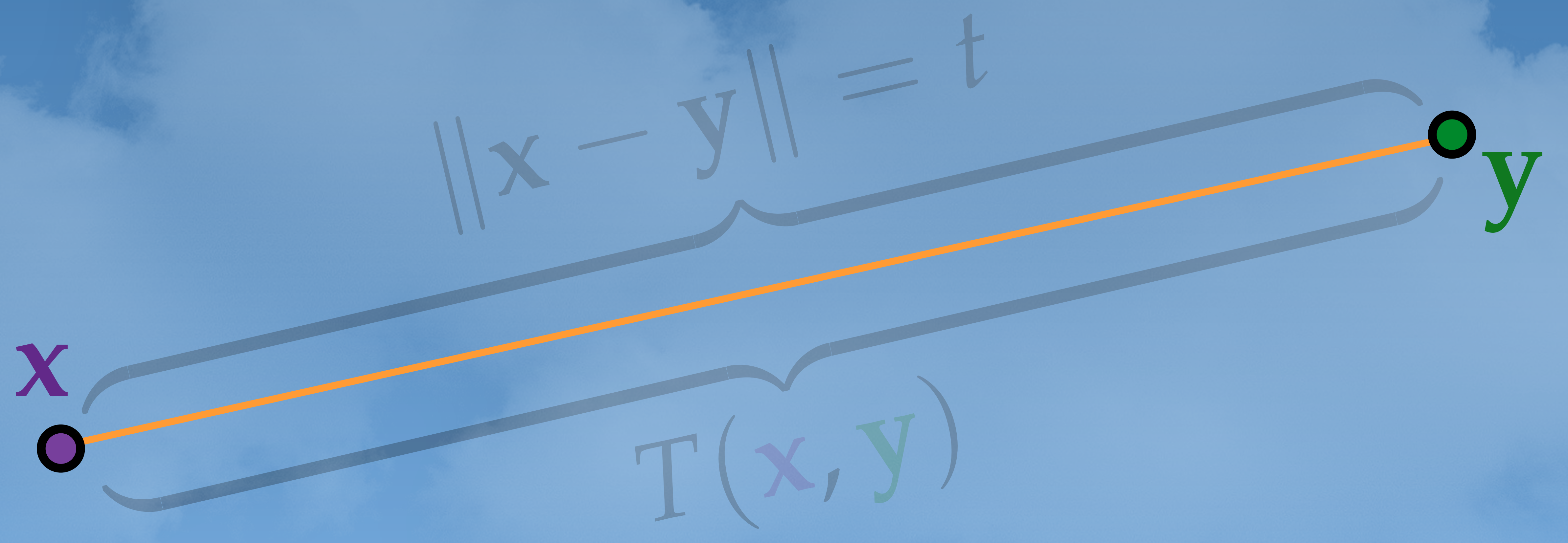
transmittance

$$\tau(\mathbf{x}, \mathbf{y}) = \int_0^t \mu_{\mathrm{t}}(\mathbf{x} - s\omega)\, \mathrm{d}s$$

"optical thickness"

Transmittance estimation

1. Estimators integrating **optical thickness**
2. Estimators using **free-flight sampling**



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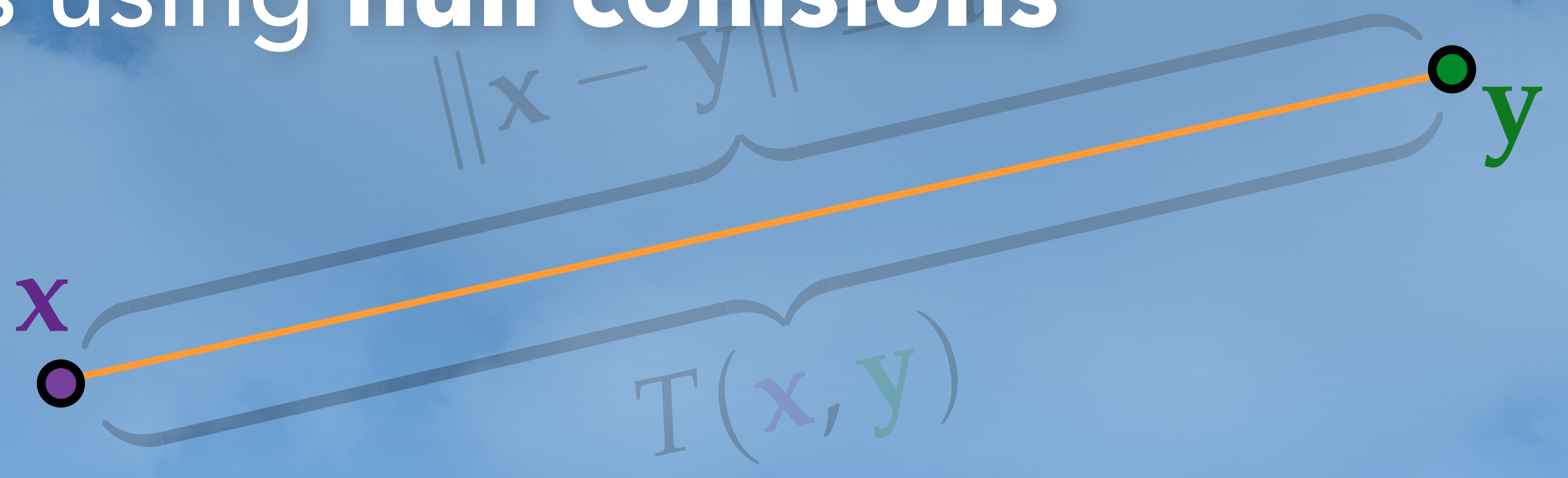
transmittance

$$\tau(\mathbf{x},\mathbf{y}) = \int_0^t \mu_\mathrm{t}(\mathbf{x} - s\omega)\,\mathrm{d}s$$

"optical thickness"

Transmittance estimation

1. Estimators integrating **optical thickness**
2. Estimators using **free-flight sampling**
3. Estimators using **null collisions**



$$T(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{-\tau(\mathbf{x}, \mathbf{y})}$$

transmittance

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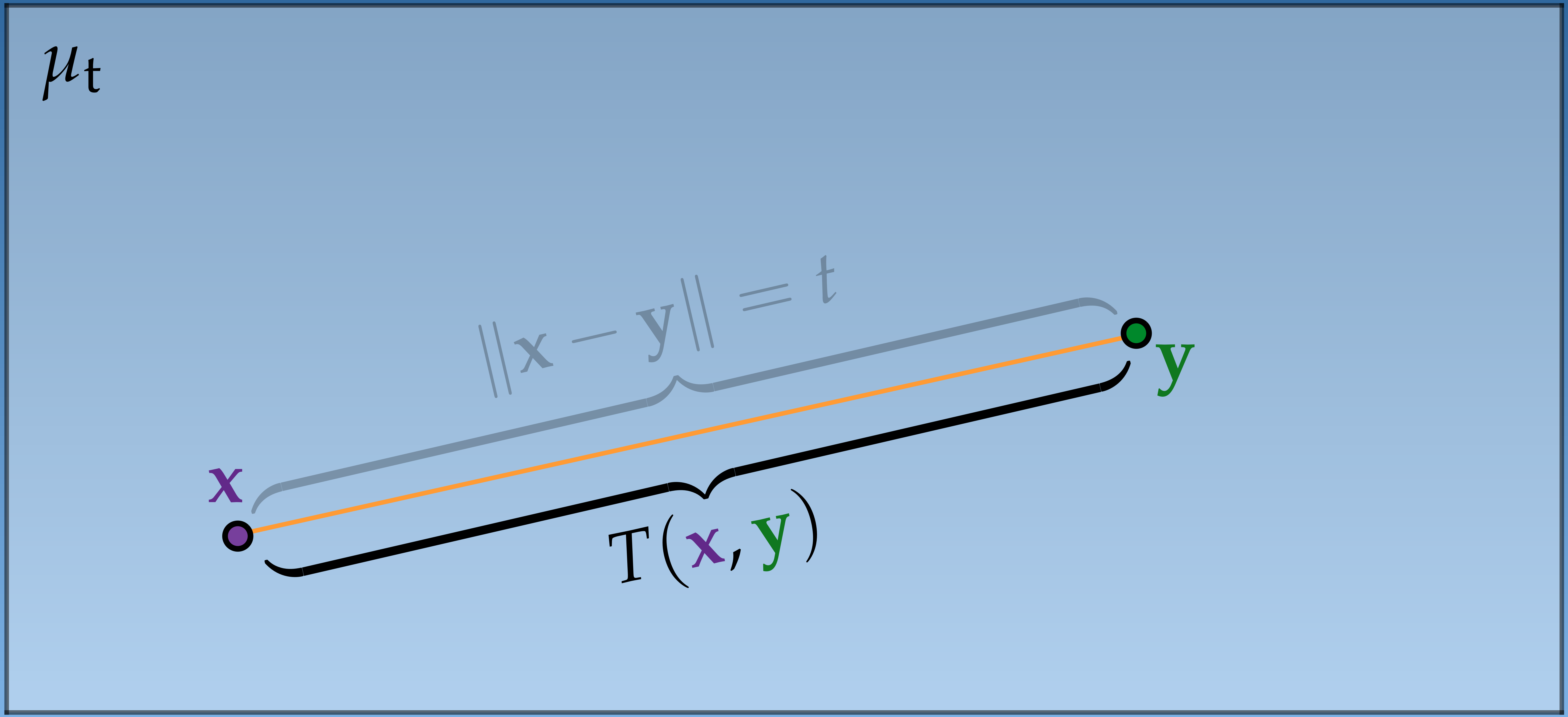
"optical thickness"

Homogeneous medium



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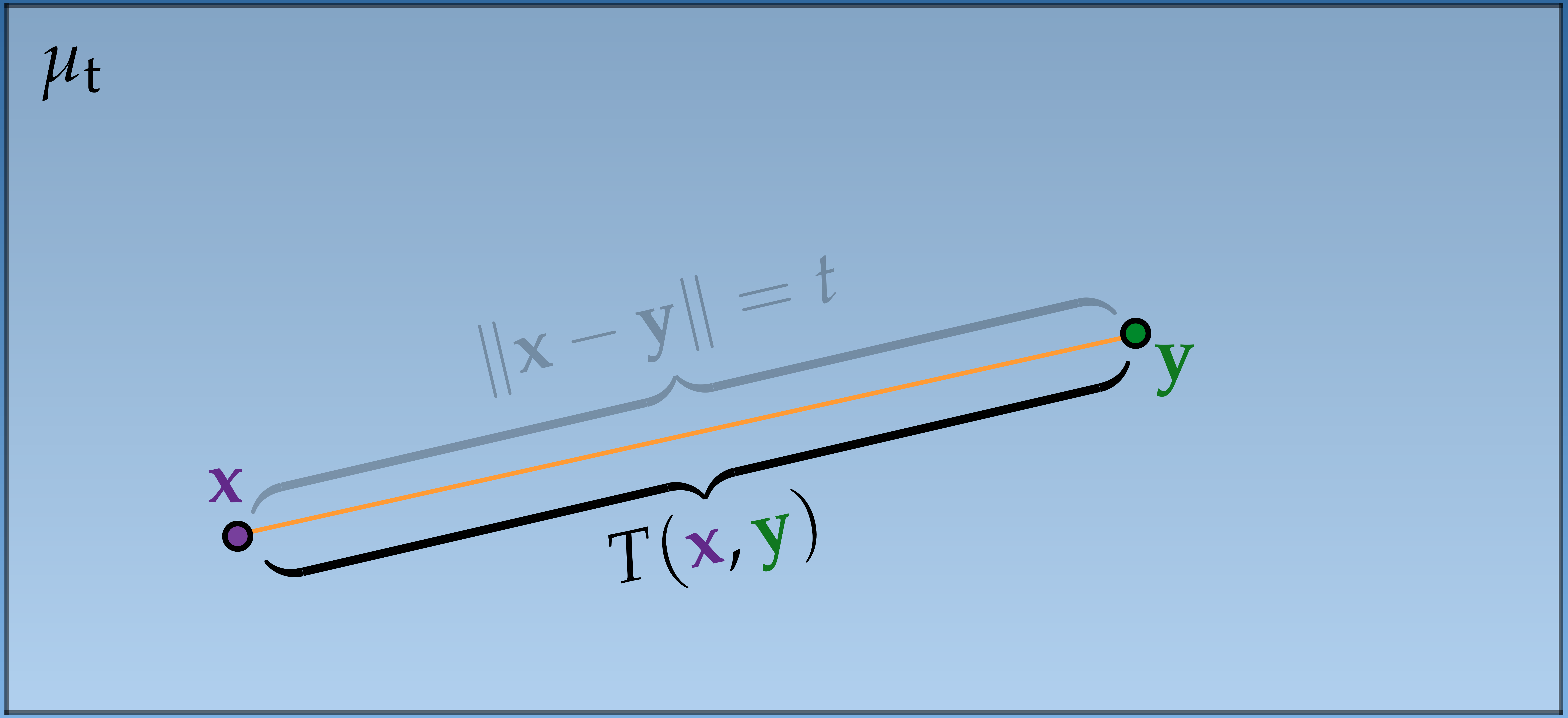
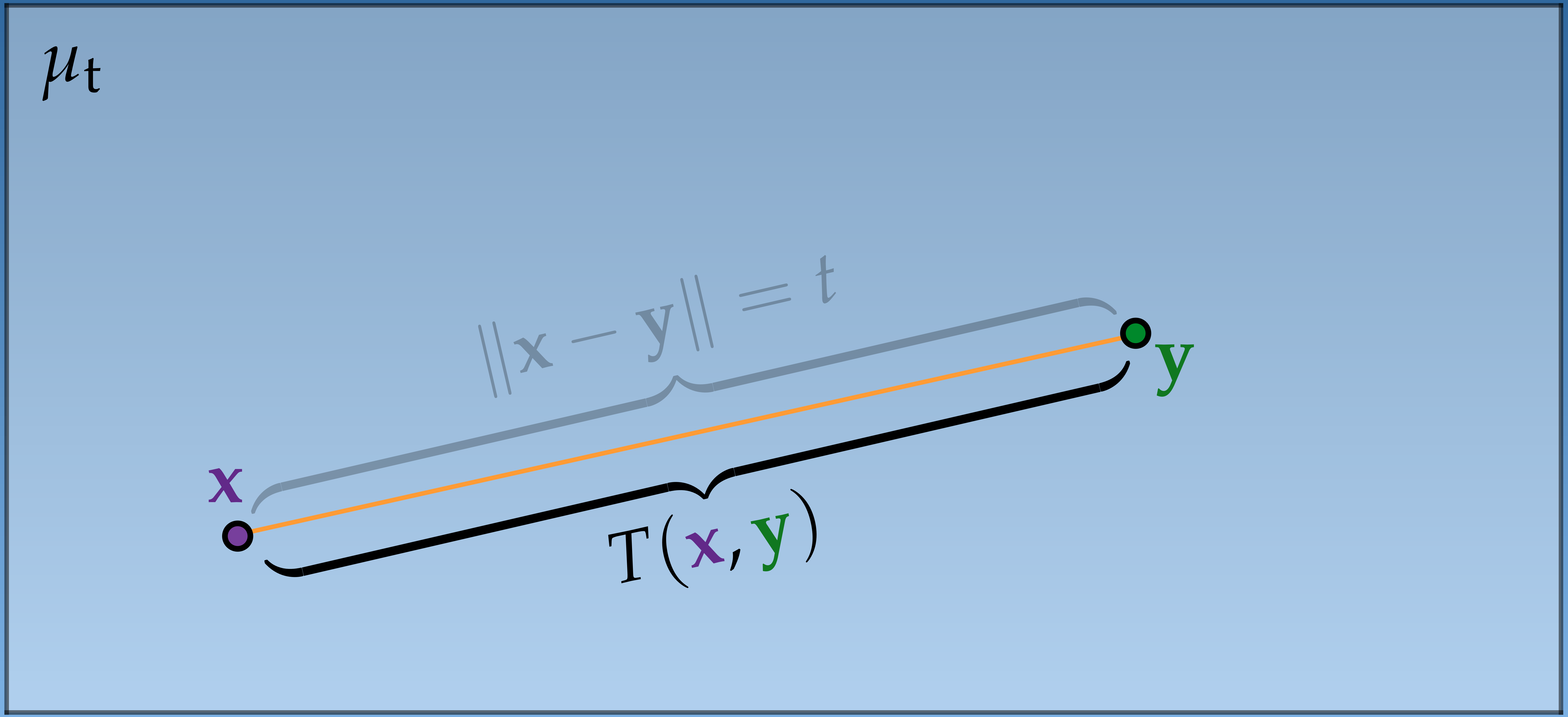


$$T(\mathbf{x},\mathbf{y}) = e^{-\tau(\mathbf{x},\mathbf{y})}$$

$$\tau(\mathbf{x},\mathbf{y}) = \int_0^t \mu_t(\mathbf{x}-s\omega)\,ds = \mu_t\, t$$

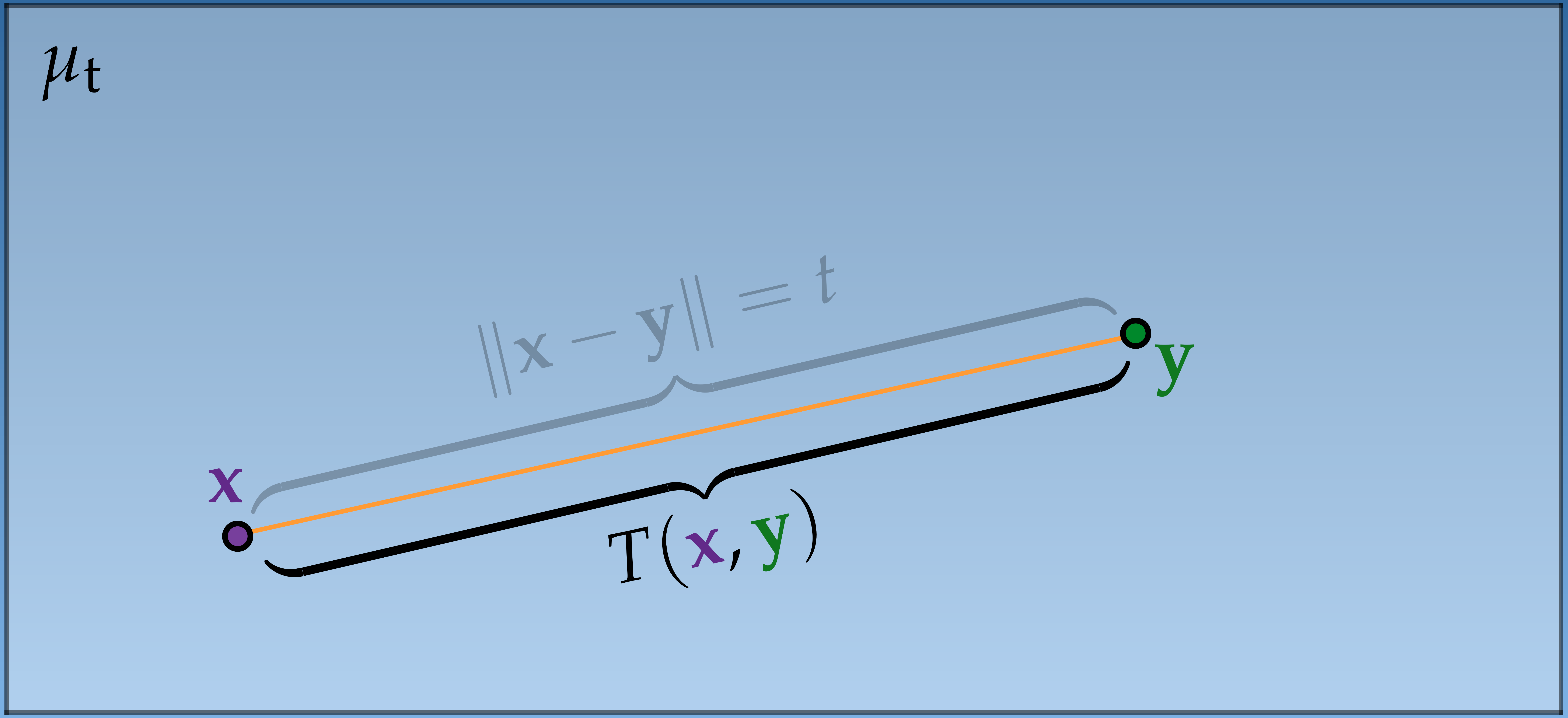
Homogeneous medium



$T(\mathbf{x}, \mathbf{y}) = e^{-}$

$\mu_t \, t$

Homogeneous medium



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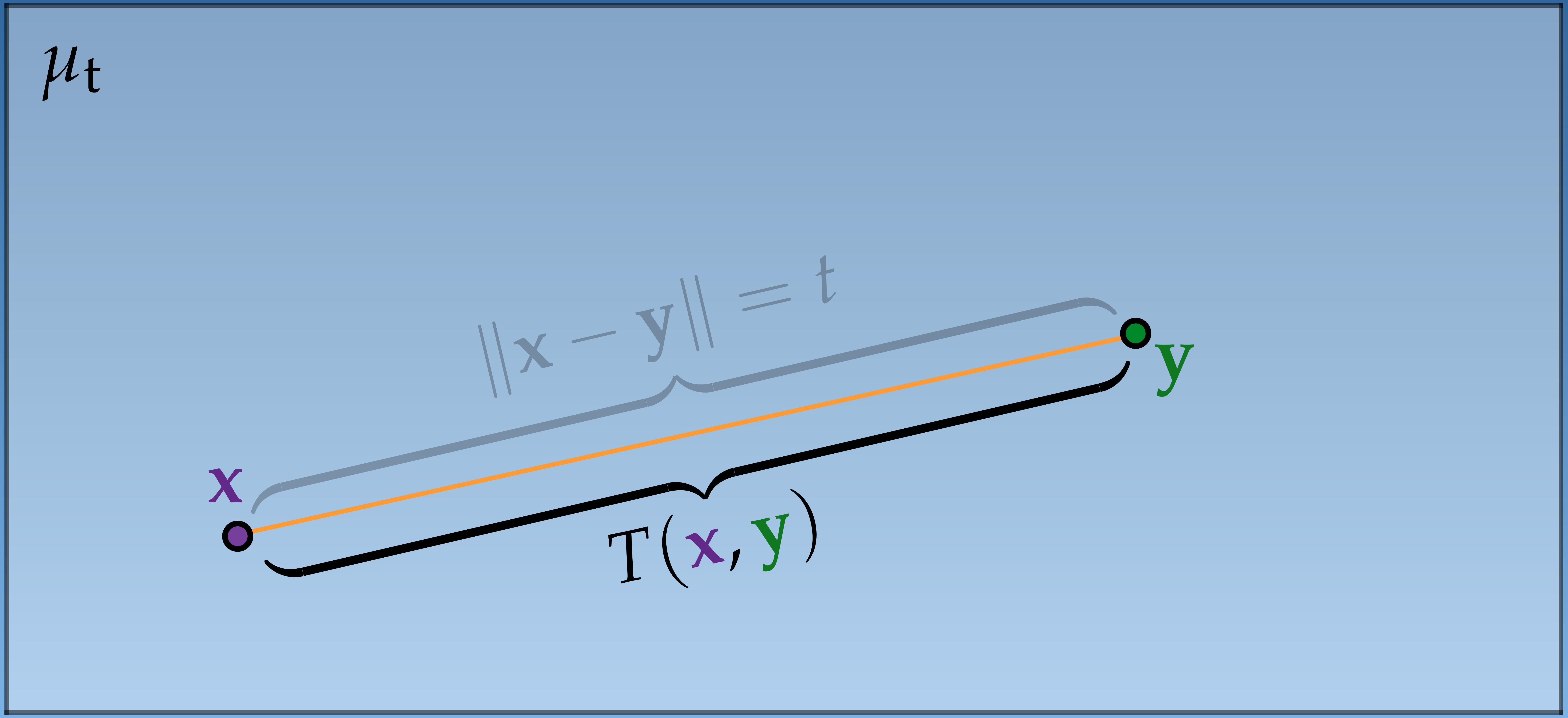
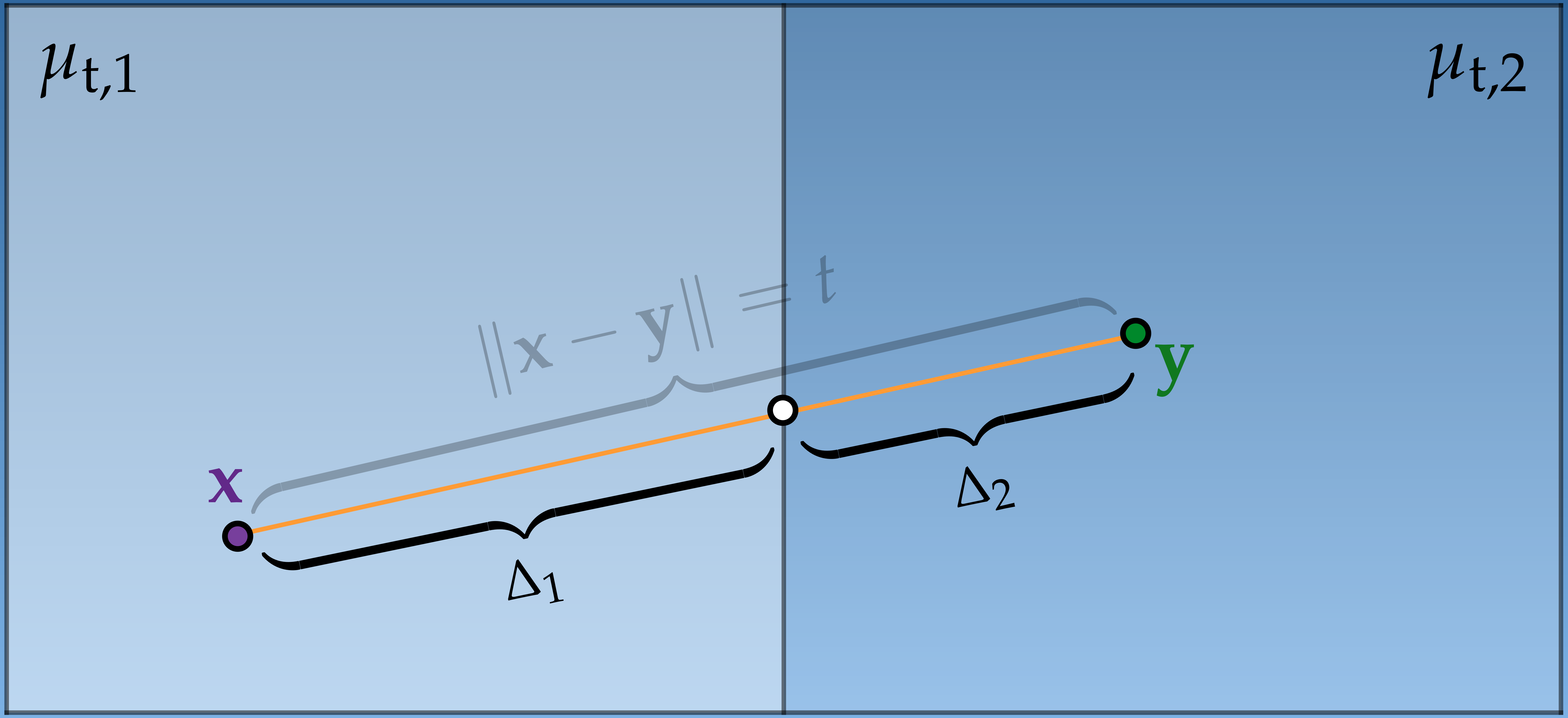


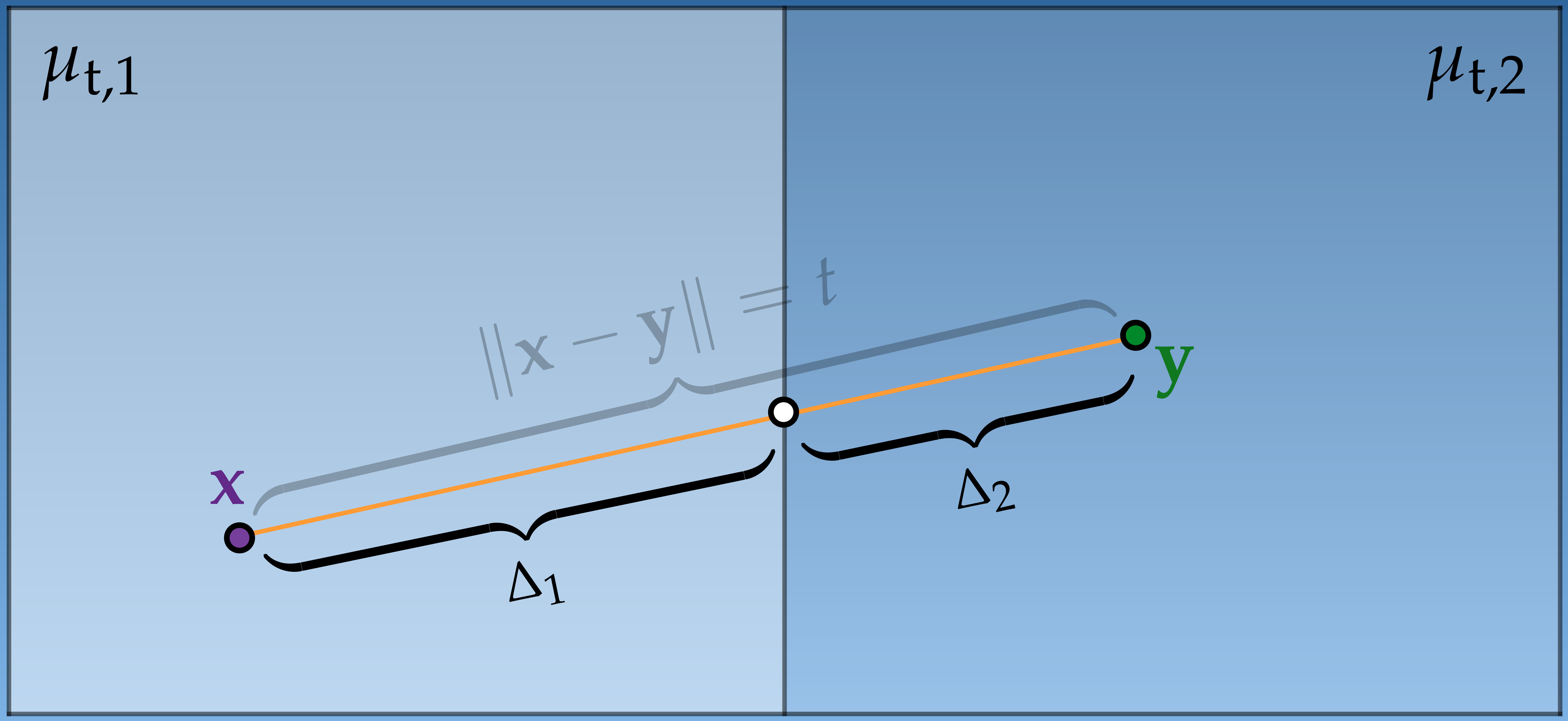
$$T(\mathbf{x}, \mathbf{y}) = e^{-\mu_t t}$$

$\langle T(t) \rangle_{EV}$: "**E**xpected **V**alue" estimator

Piecewise homogeneous medium

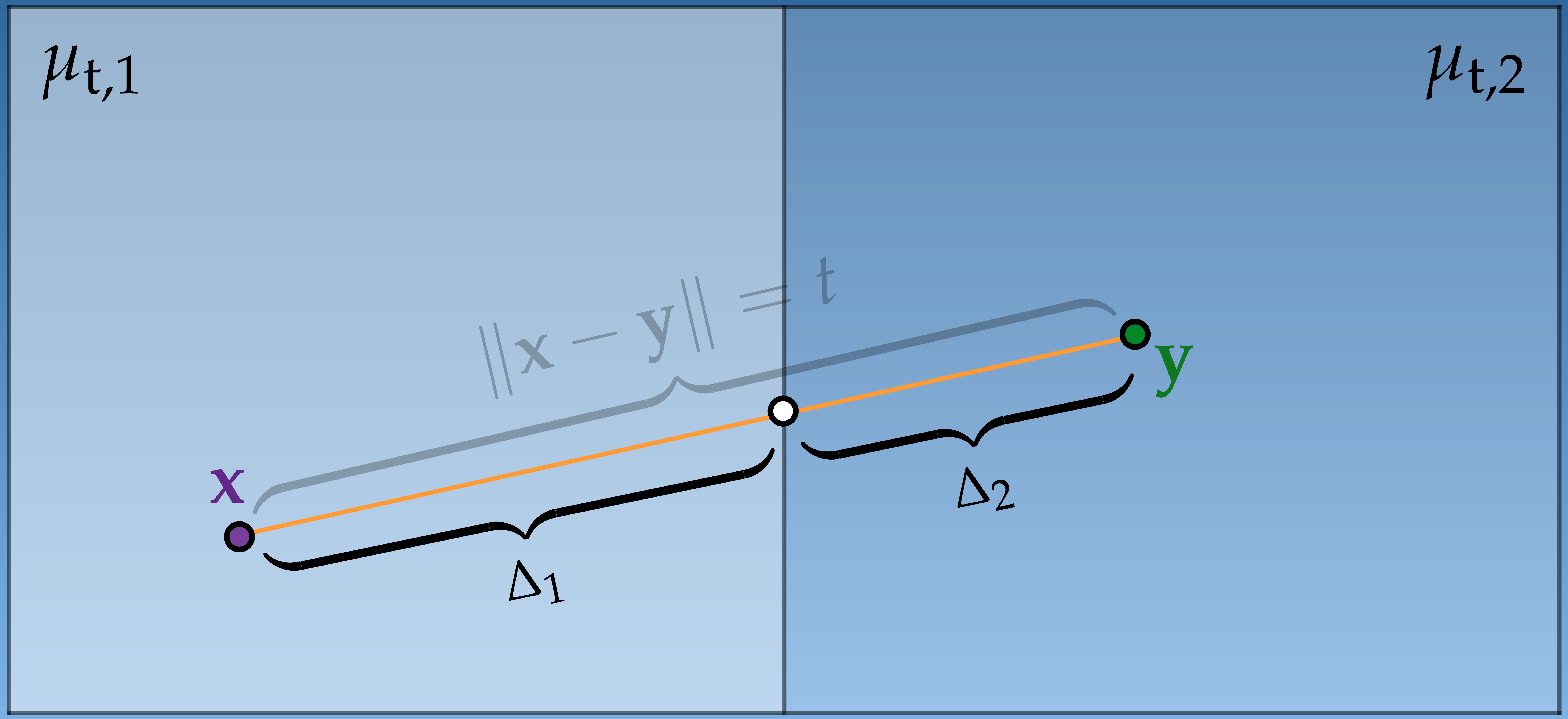


Piecewise homogeneous medium



$$t = \Delta_1 + \Delta_2$$

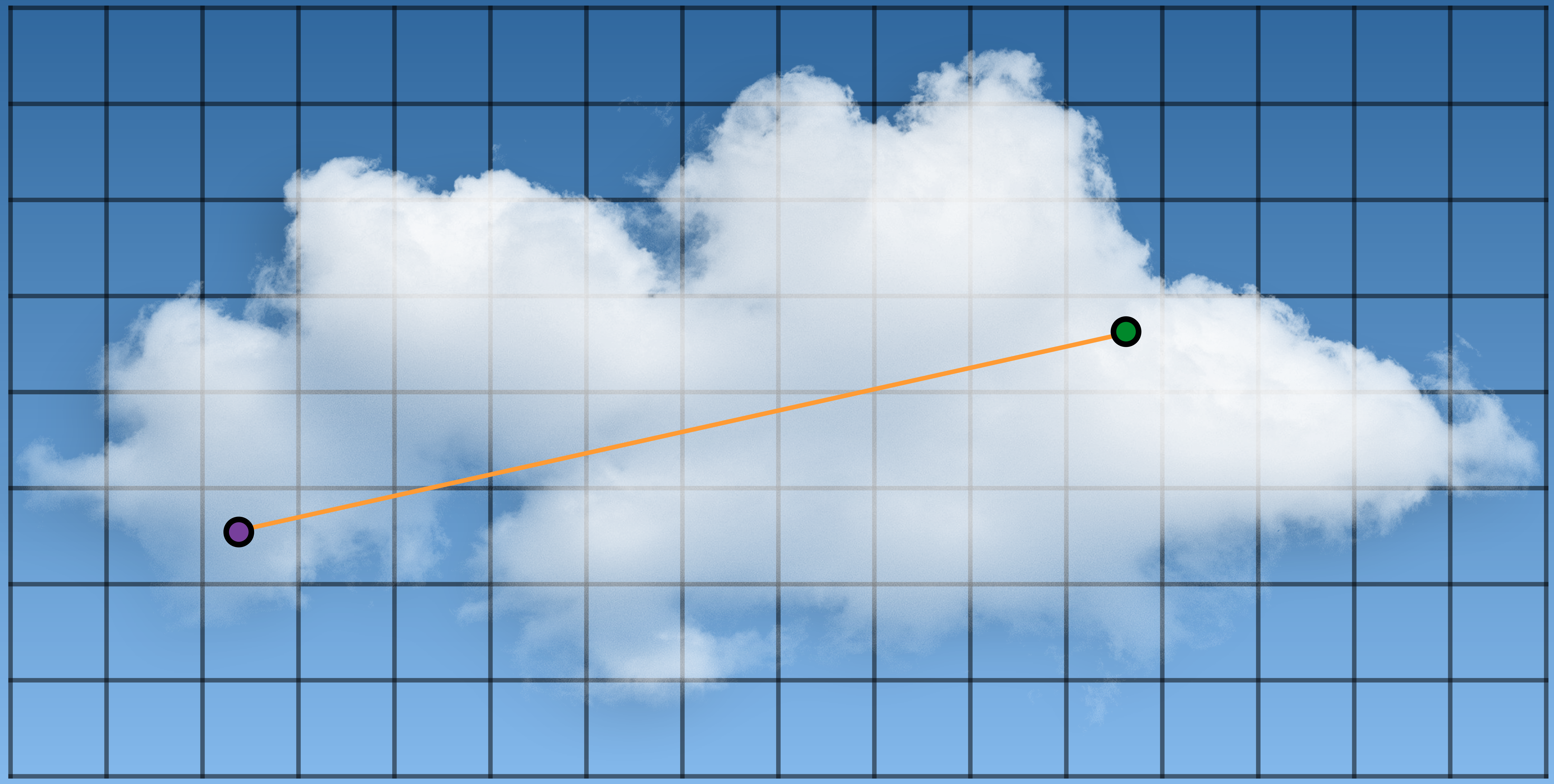
Piecewise homogeneous medium



$$t = \Delta_1 + \Delta_2$$

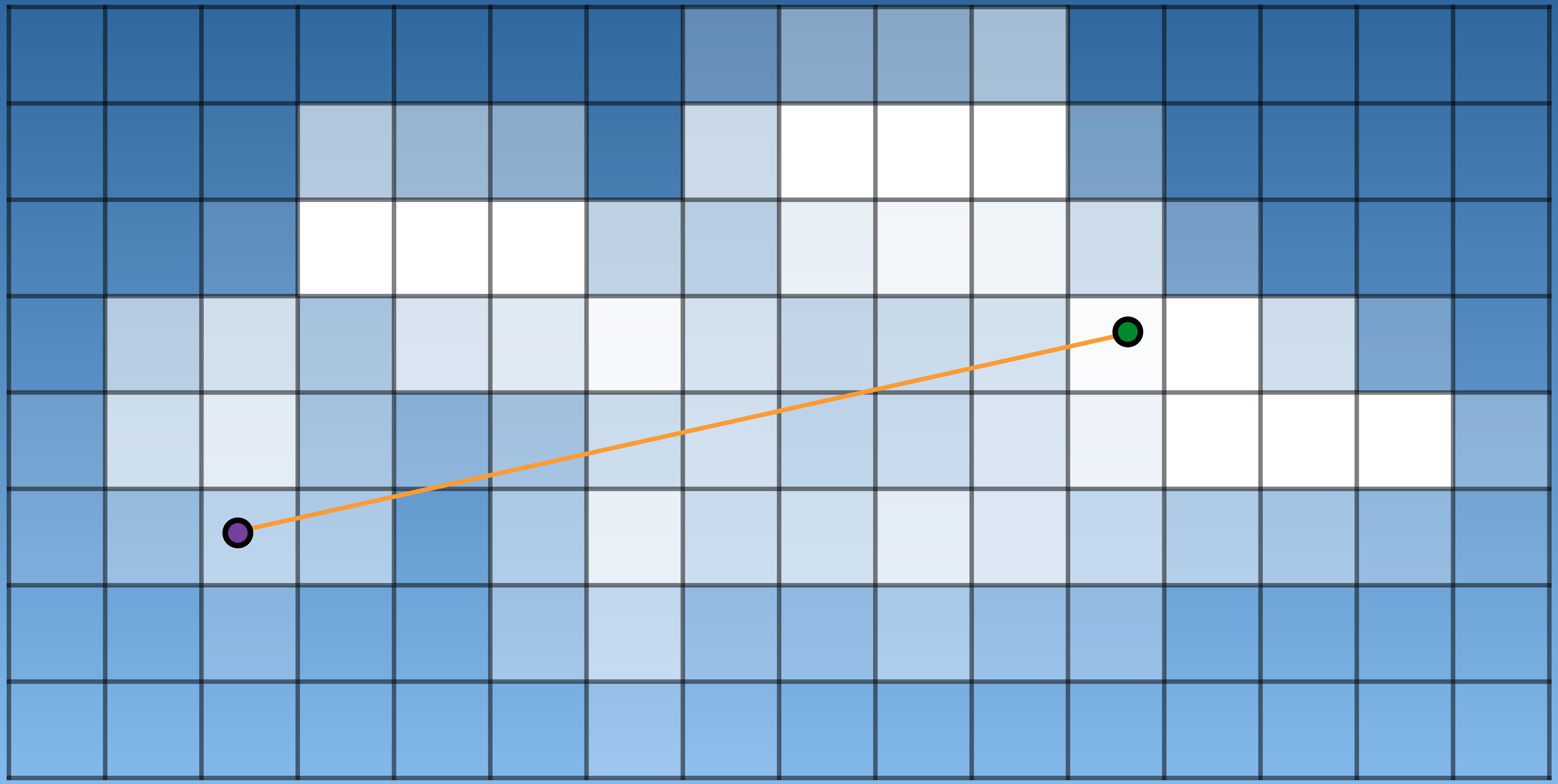
$$T(t) = e^{-\tau(t)} = e^{-(\tau_1+\tau_2)} = e^{-(\mu_{t,1}\Delta_1+\mu_{t,2}\Delta_2)}$$

Voxelize medium



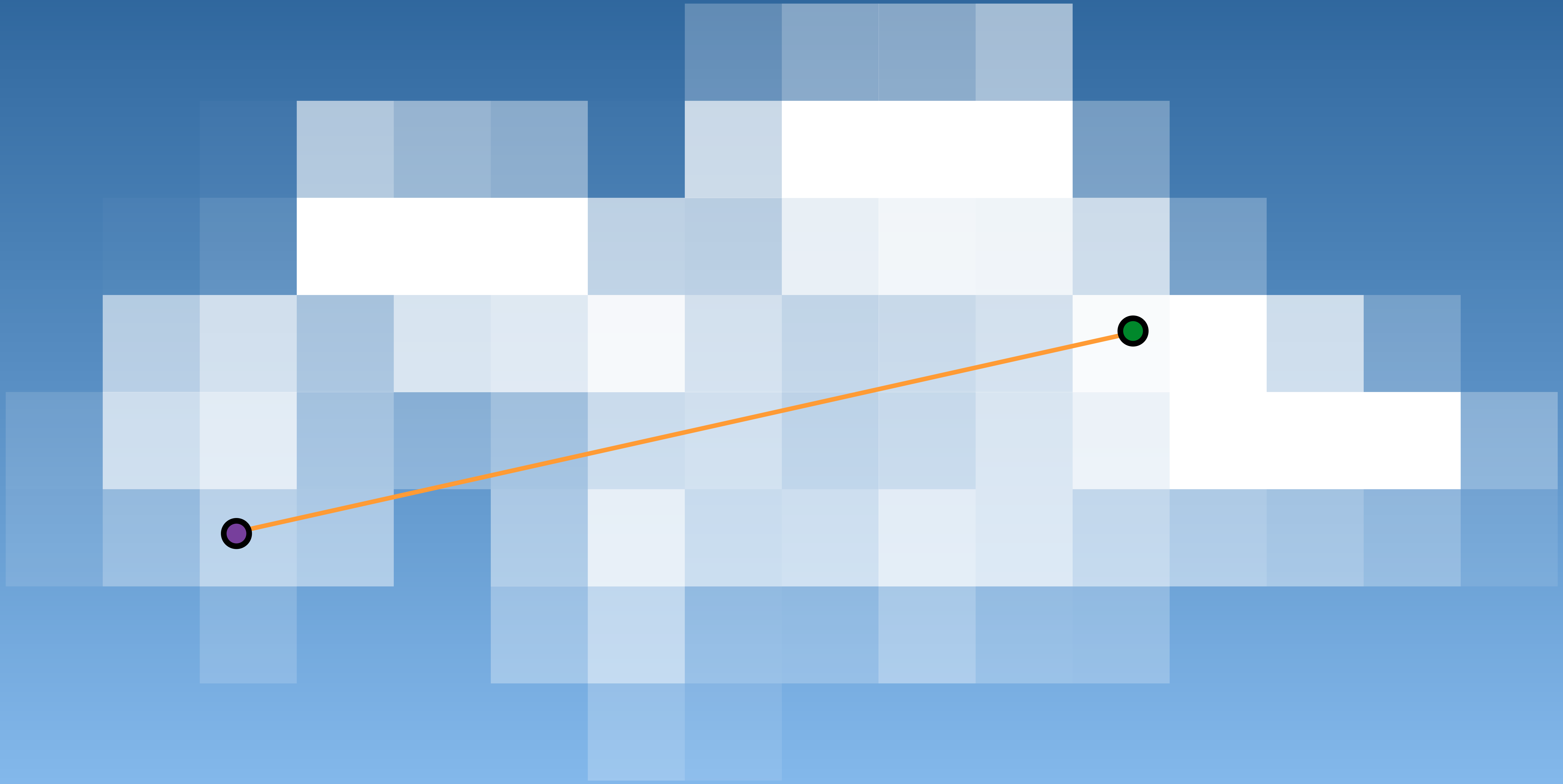
$$T(t) = \mathrm{e}^{-\tau(t)} = \mathrm{e}^{-(\tau_1+\tau_2)} = \mathrm{e}^{-(\mu_{\mathrm{t},1}\Delta_1+\mu_{\mathrm{t},2}\Delta_2)}$$

Voxelized medium (piecewise const.)



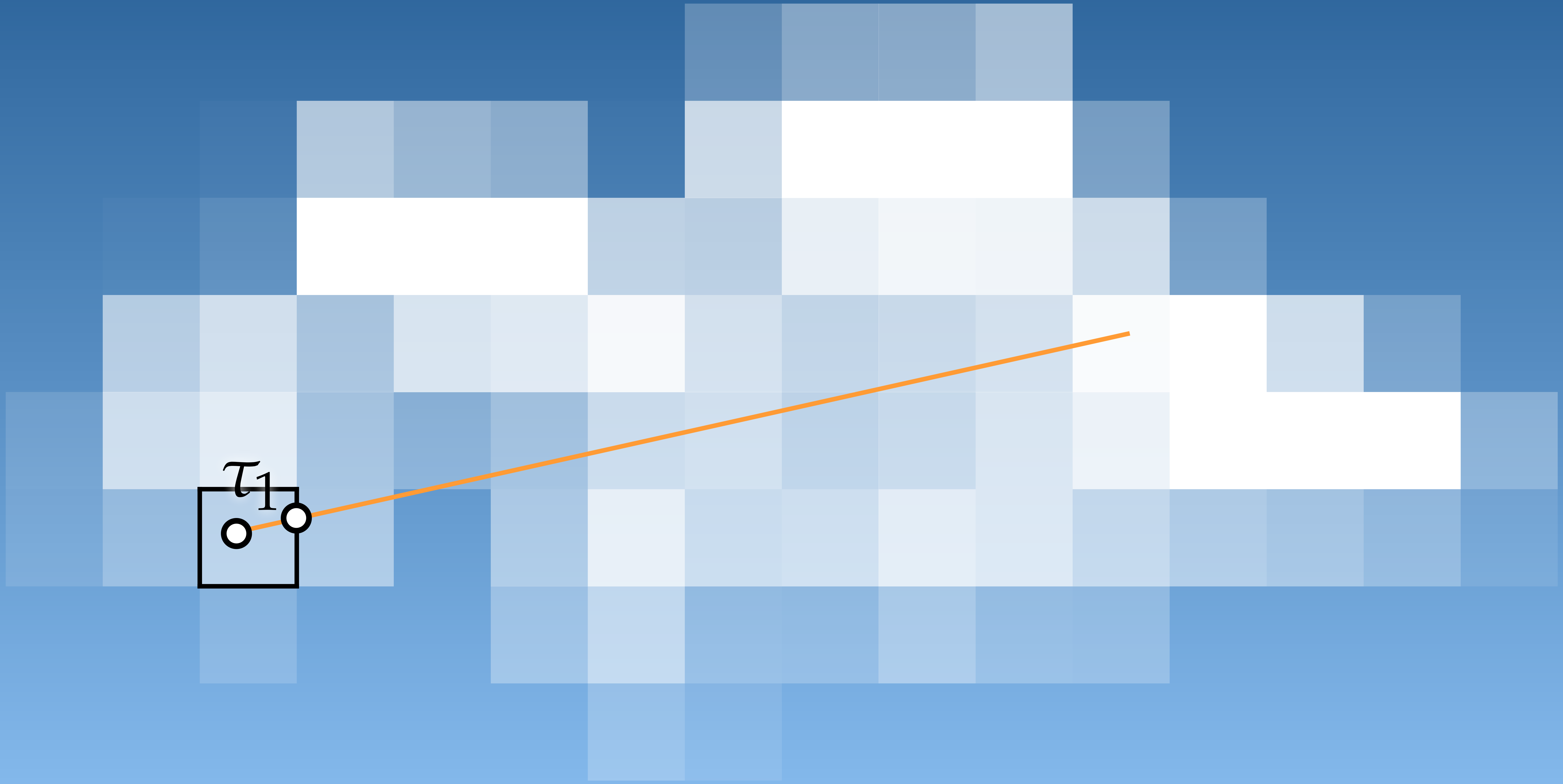
$$T(t) = \mathrm{e}^{-\tau(t)} = \mathrm{e}^{-\sum_i^k \tau_i} = \mathrm{e}^{-\sum_i^k \mu_{\mathrm{t},i}\,\Delta_i}$$

Voxelized medium (piecewise const.)



$$T(t) = e^{-\tau(t)} = e^{-\sum_i^k \tau_i} = e^{-\sum_i^k \mu_{t,i}\,\Delta_i}$$

Regular tracking (piecewise constant)



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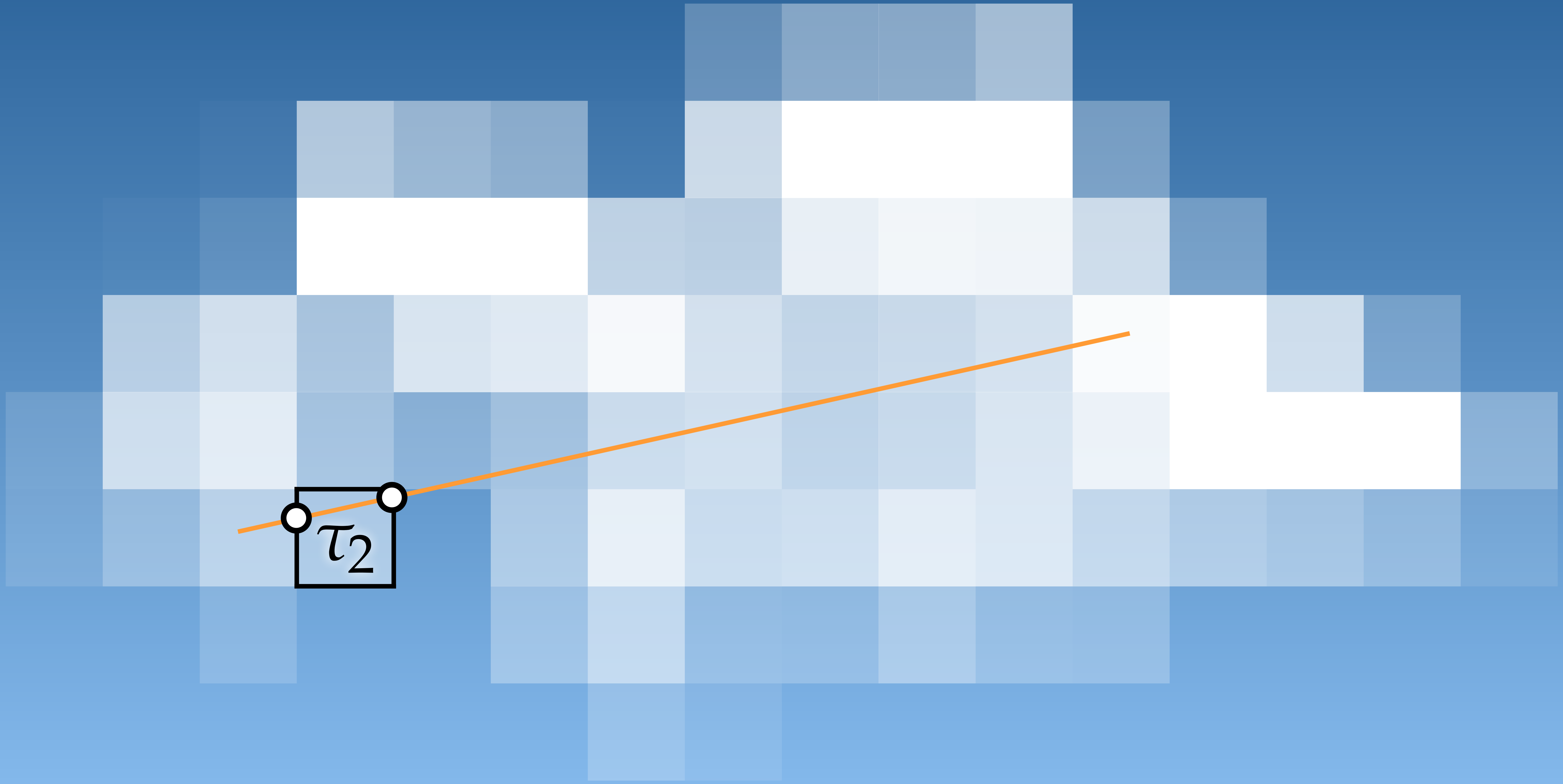

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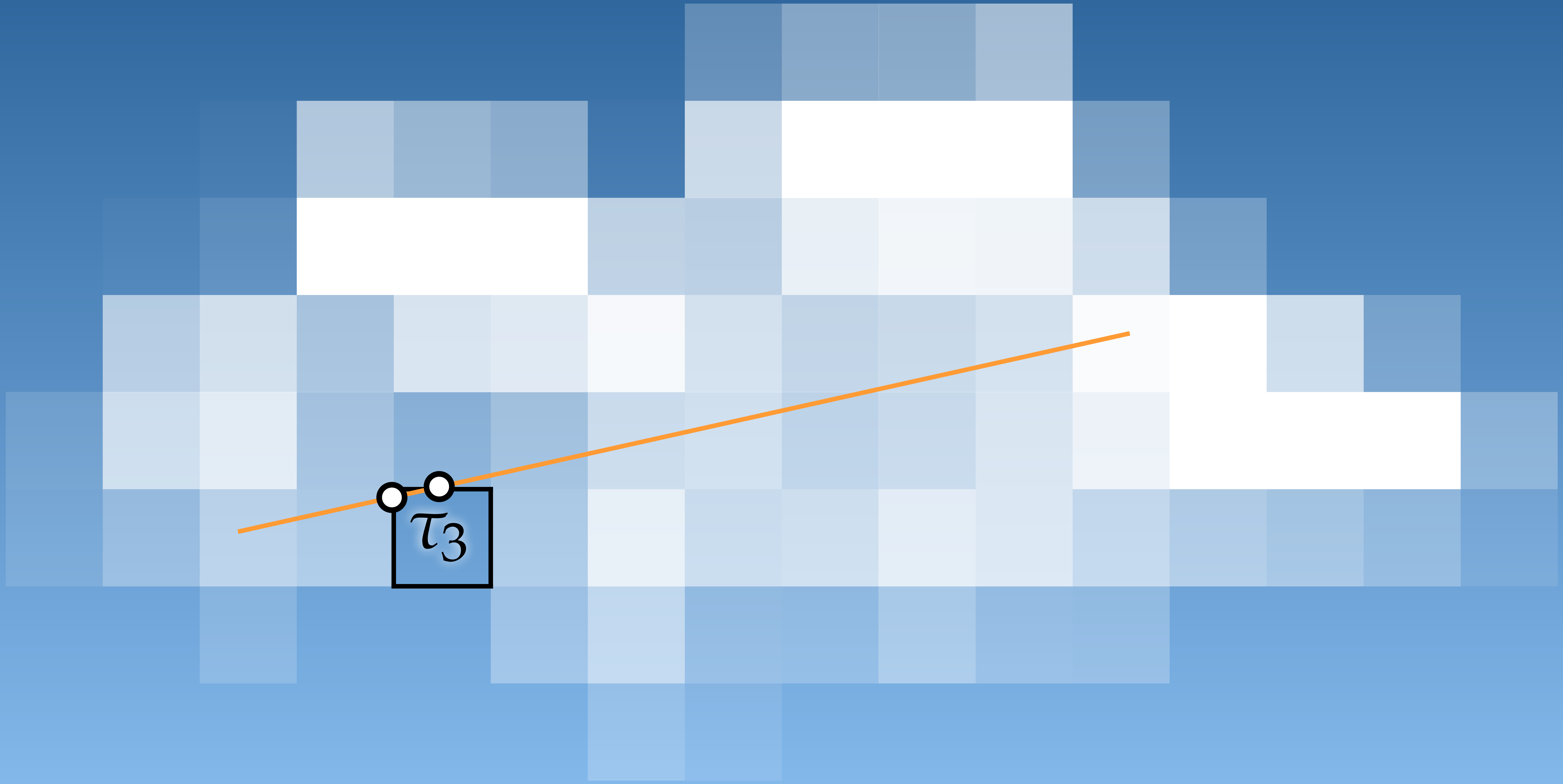

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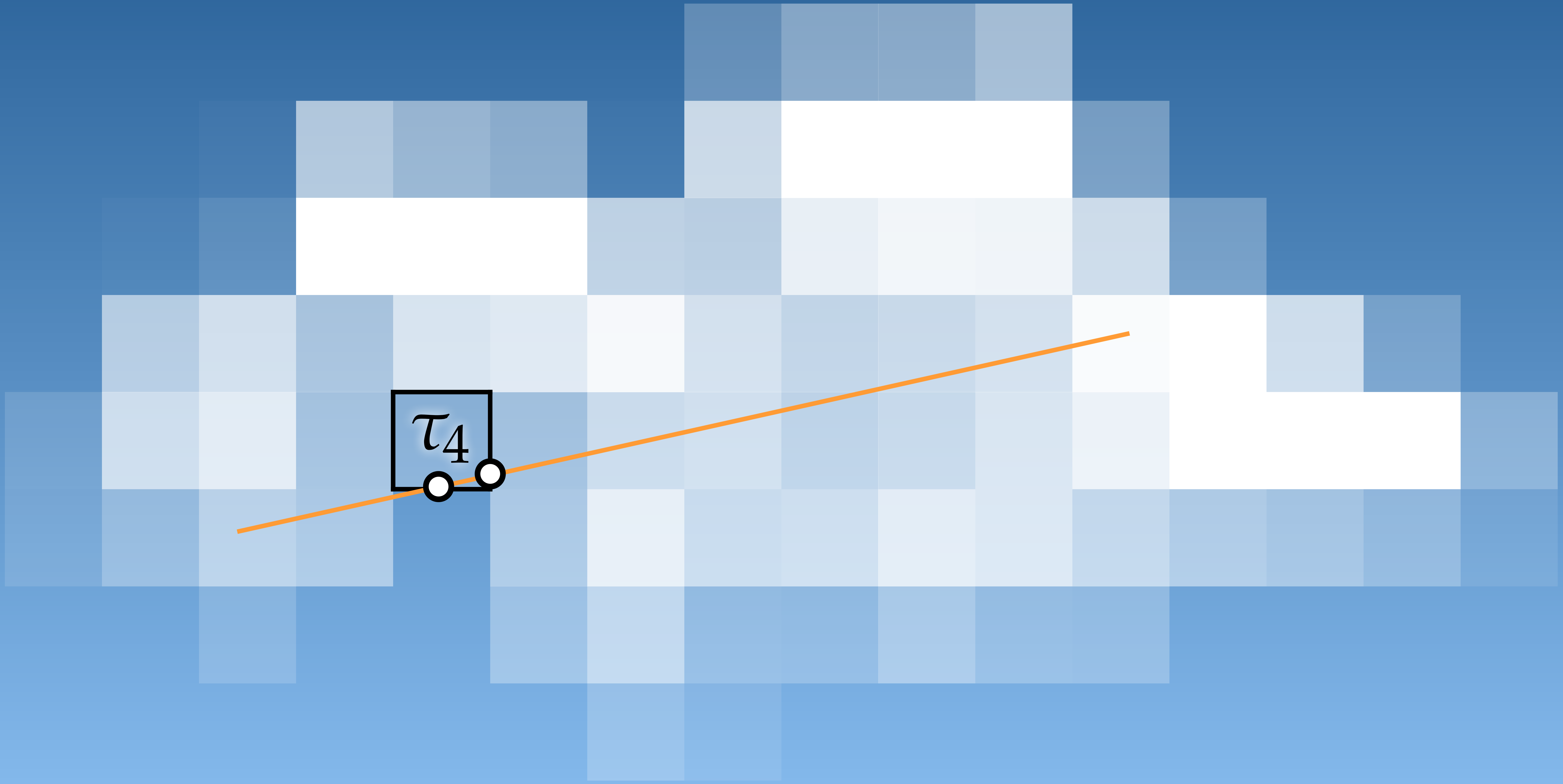

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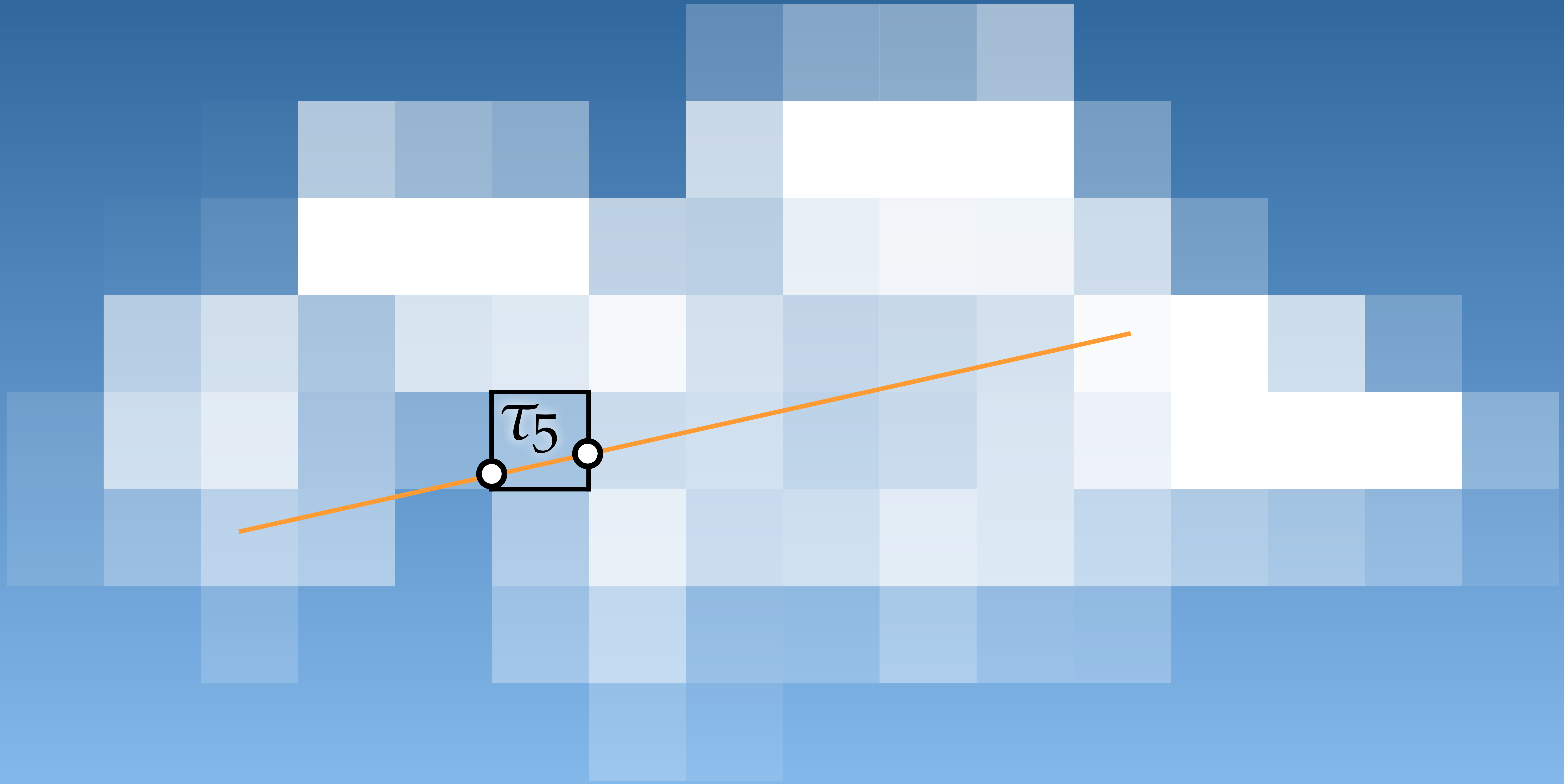

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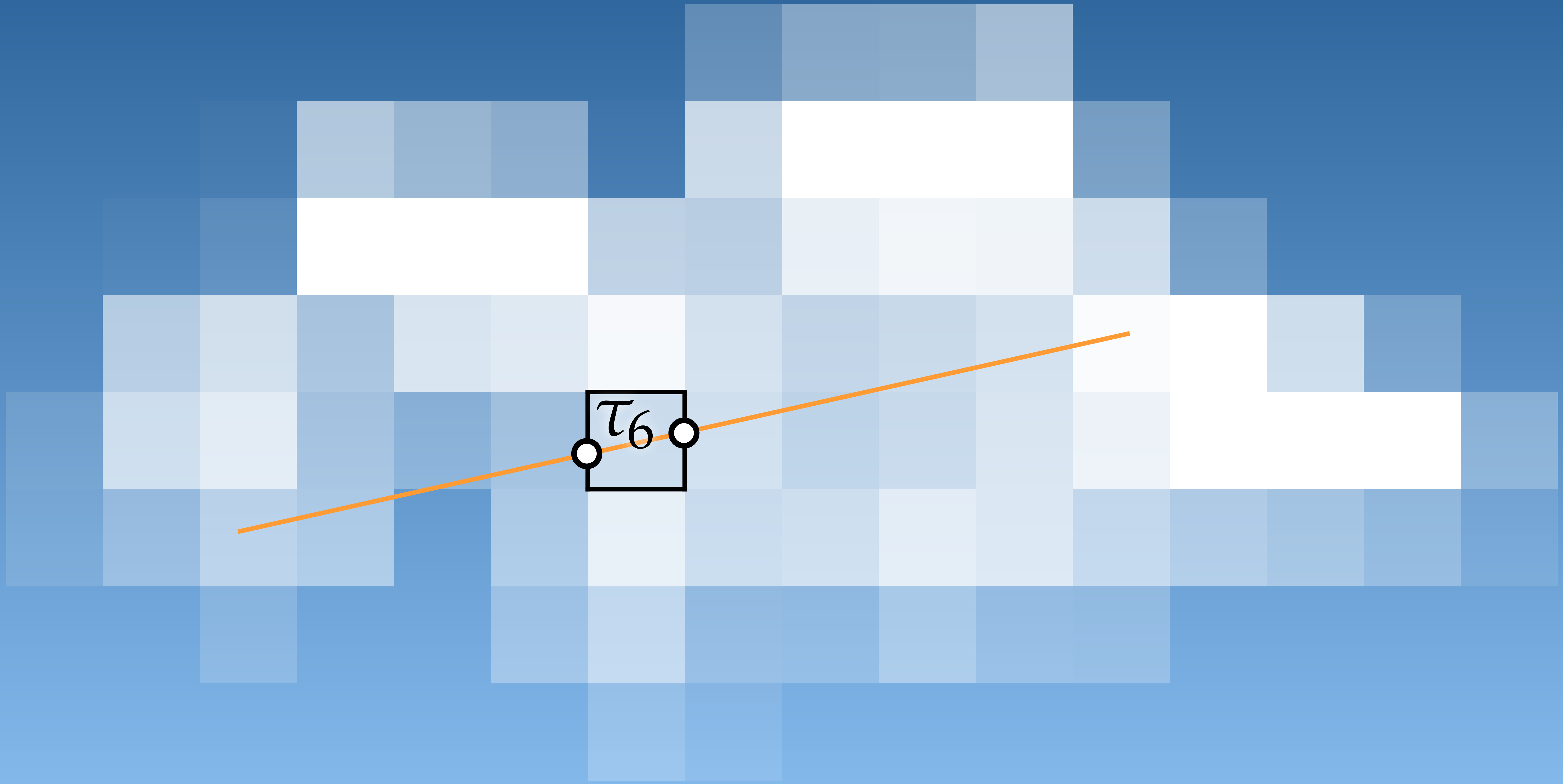

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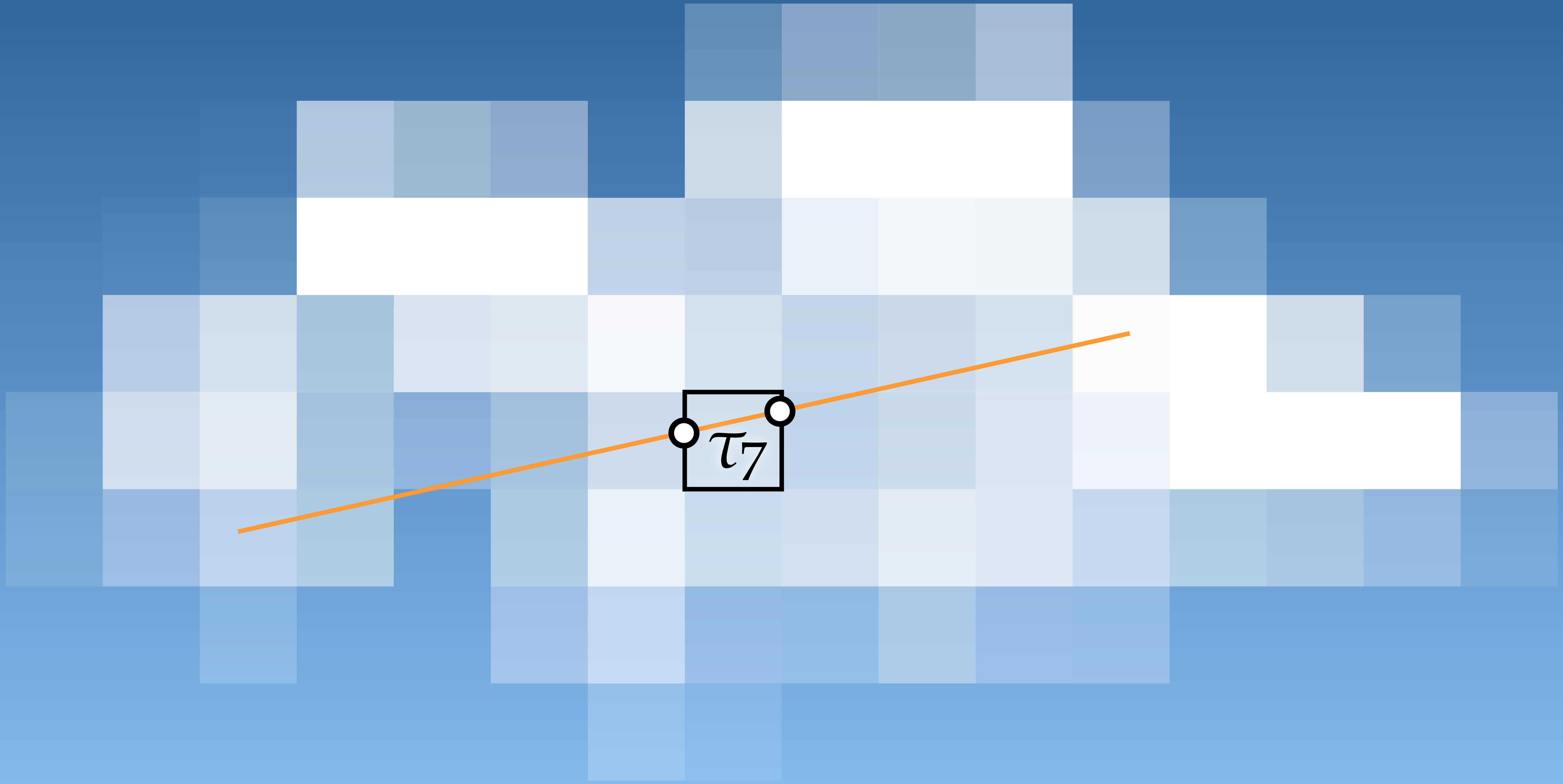

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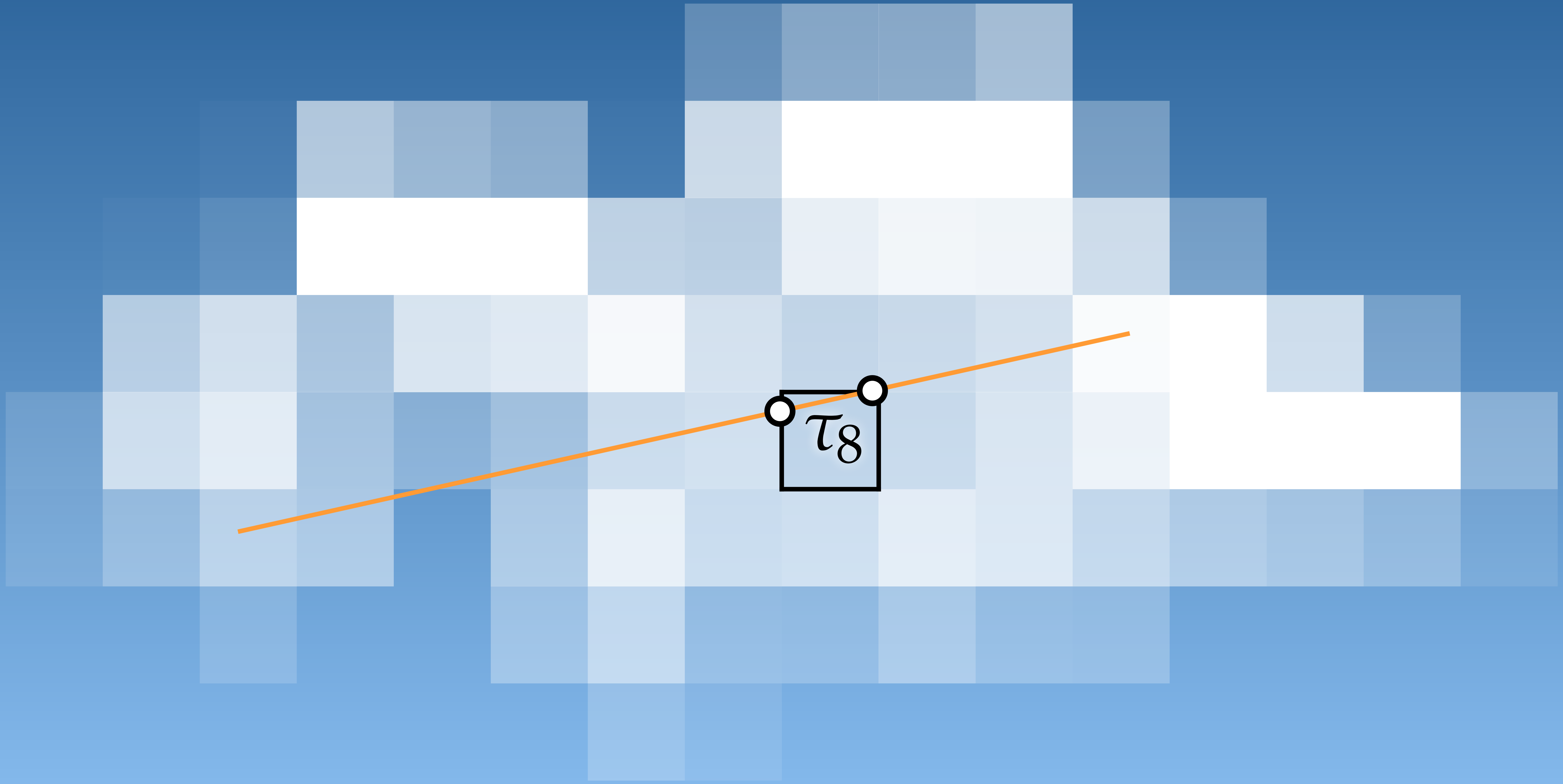

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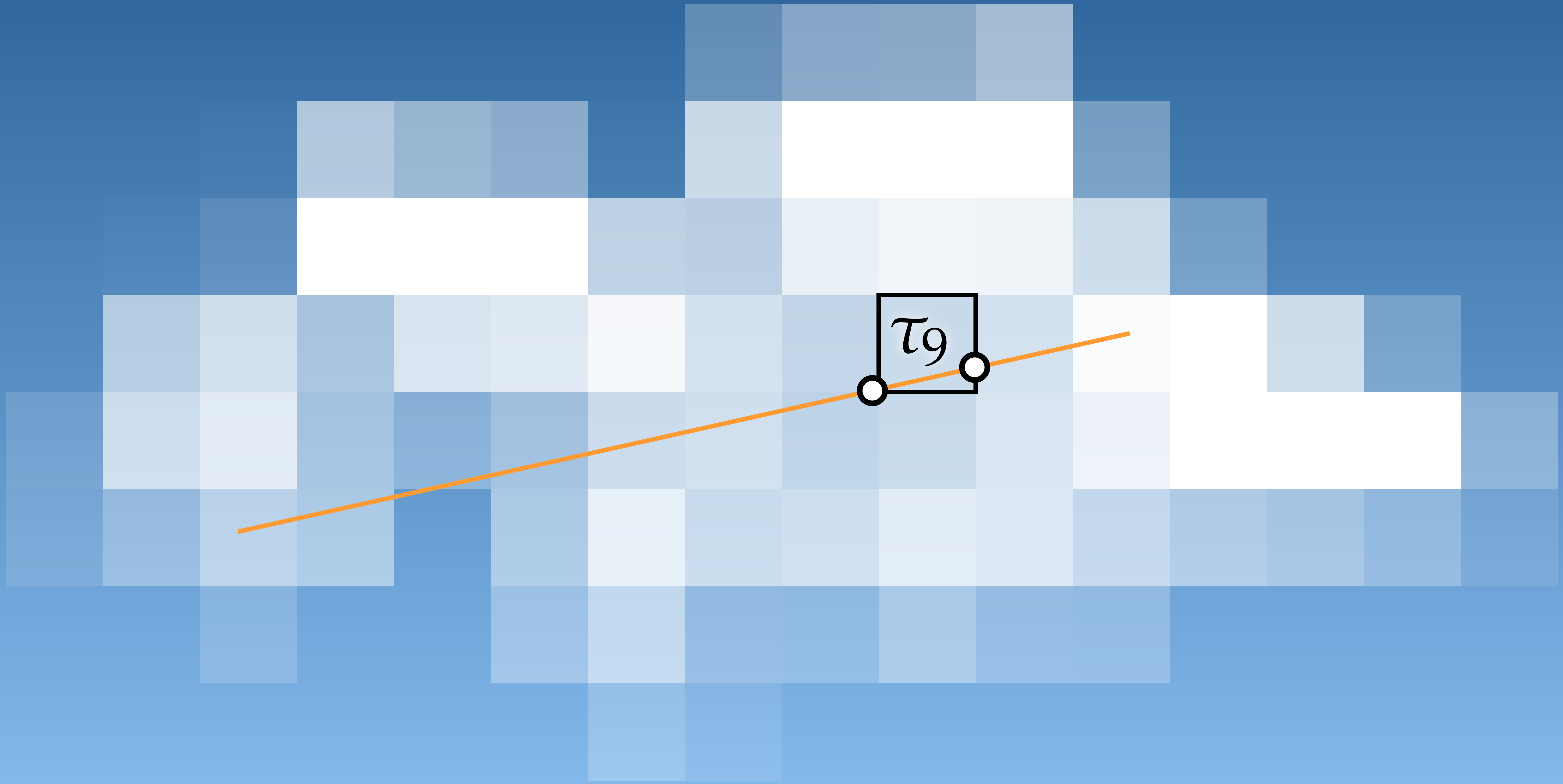

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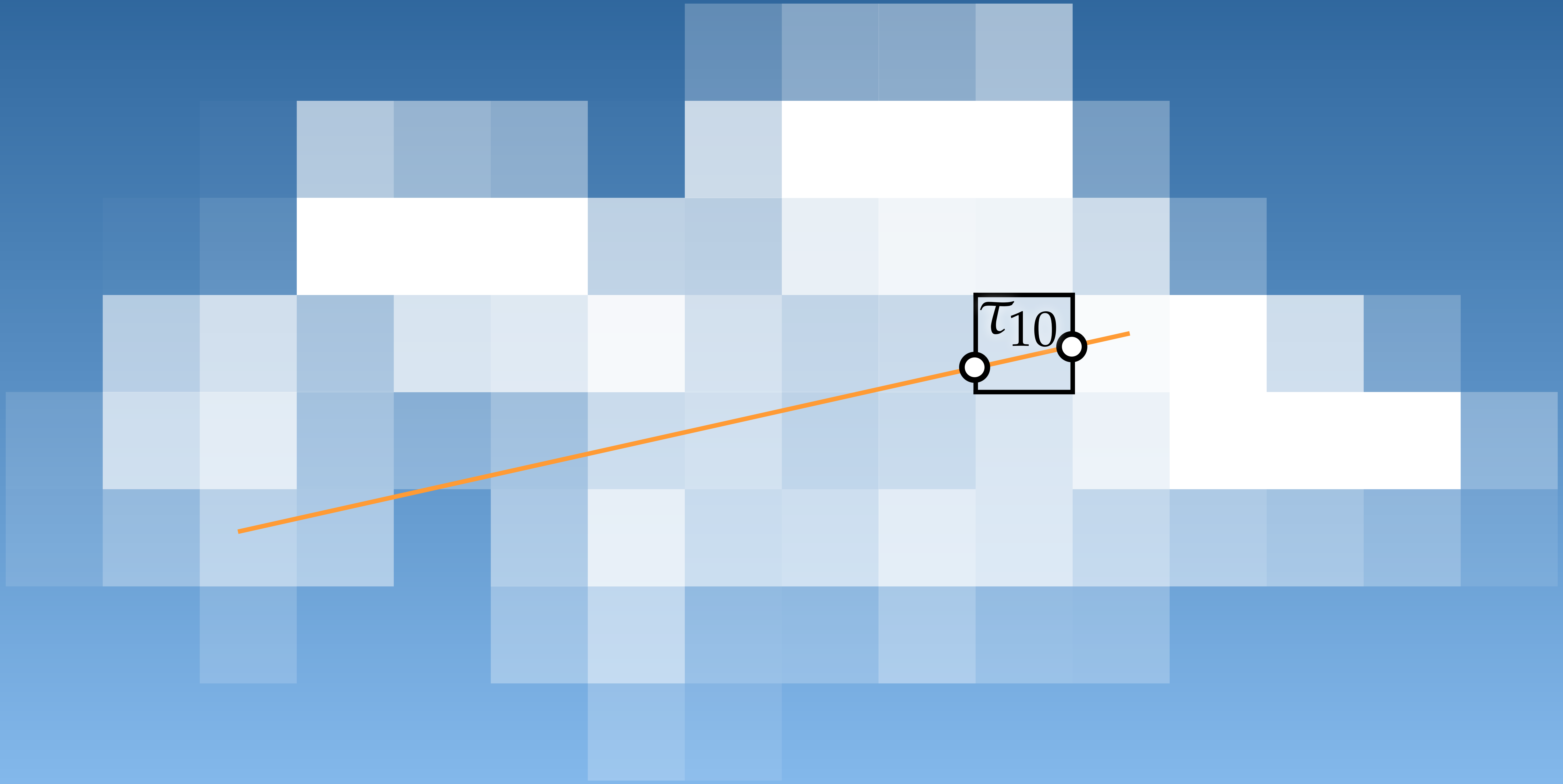

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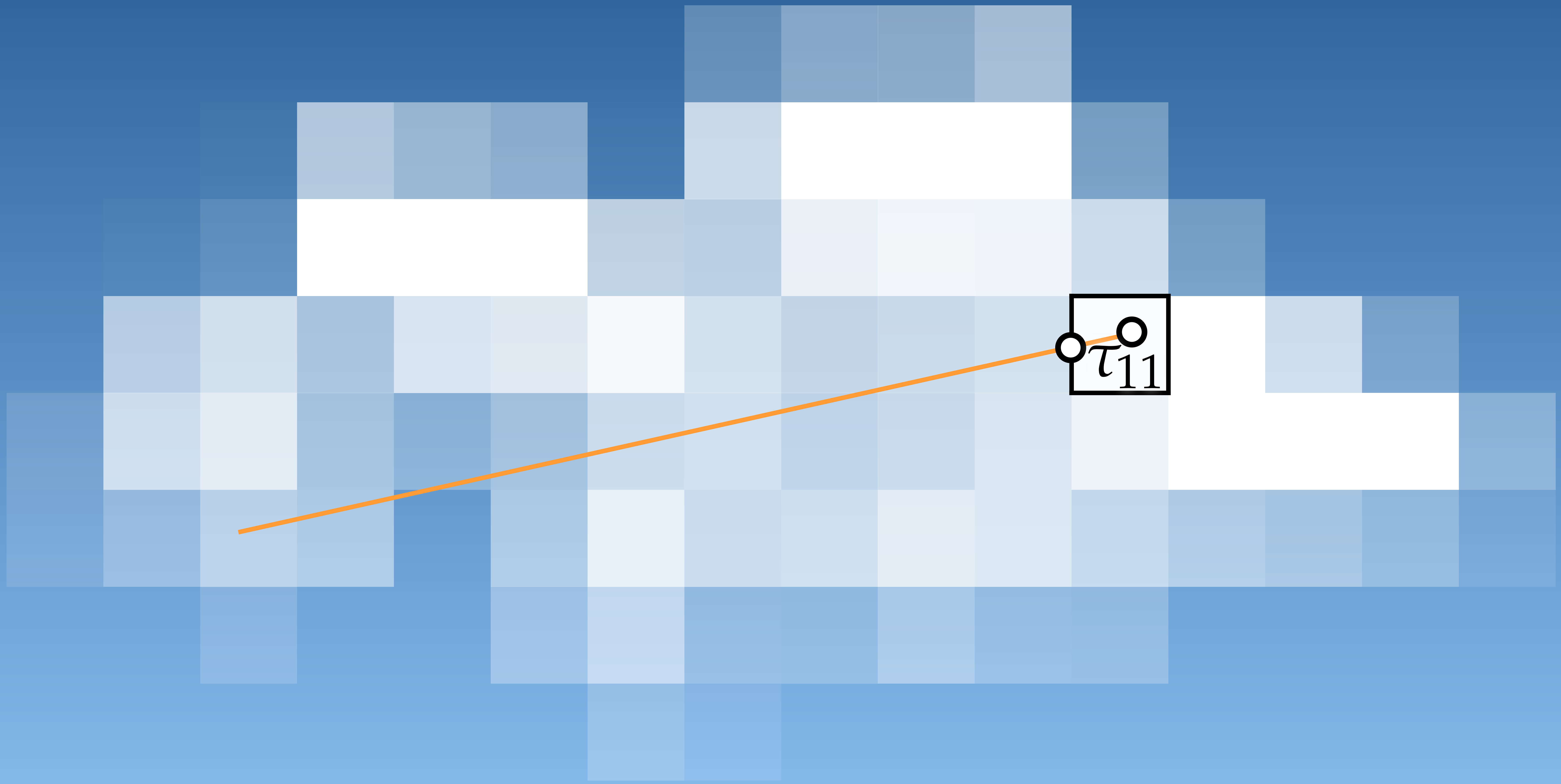

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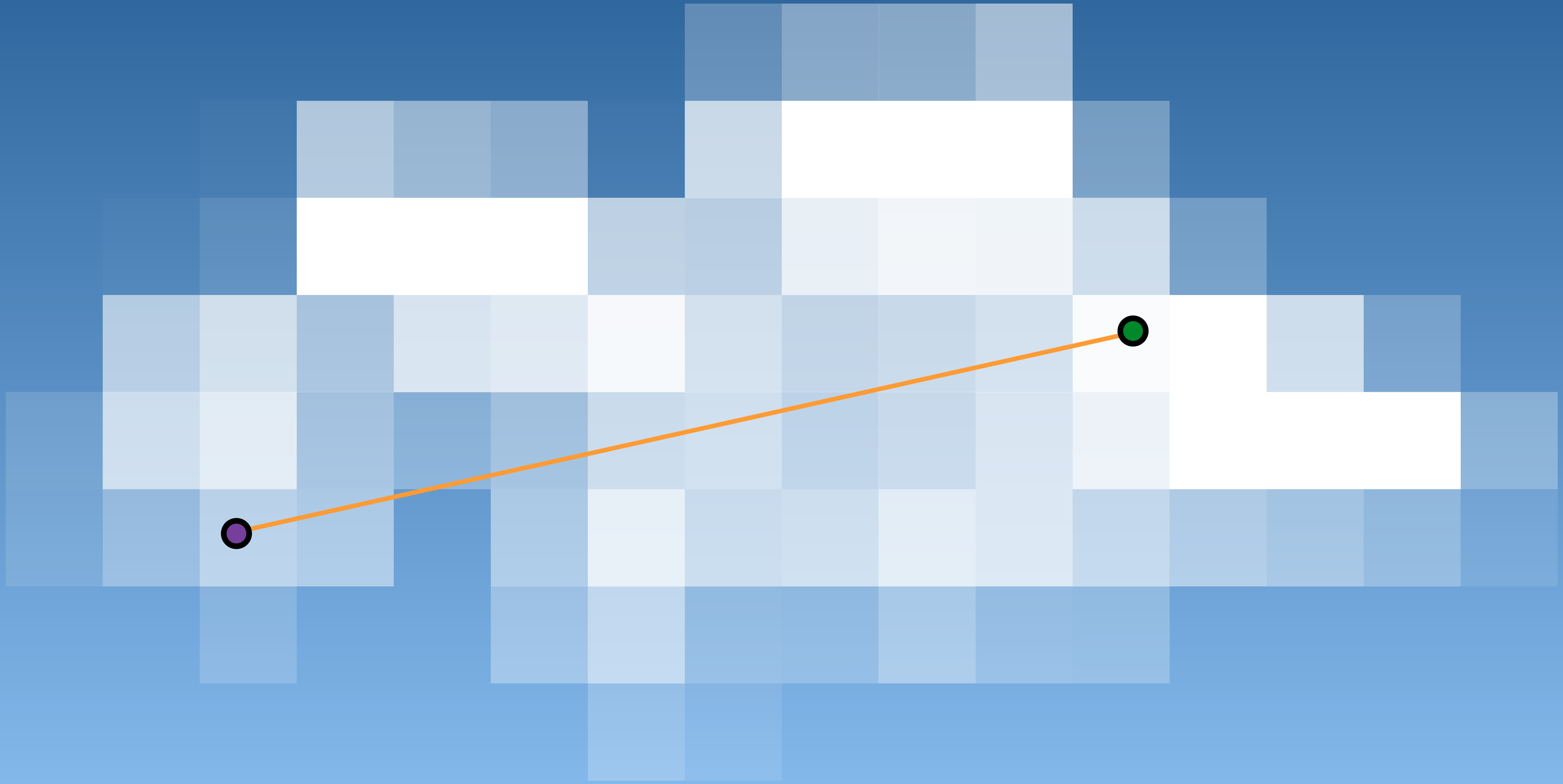

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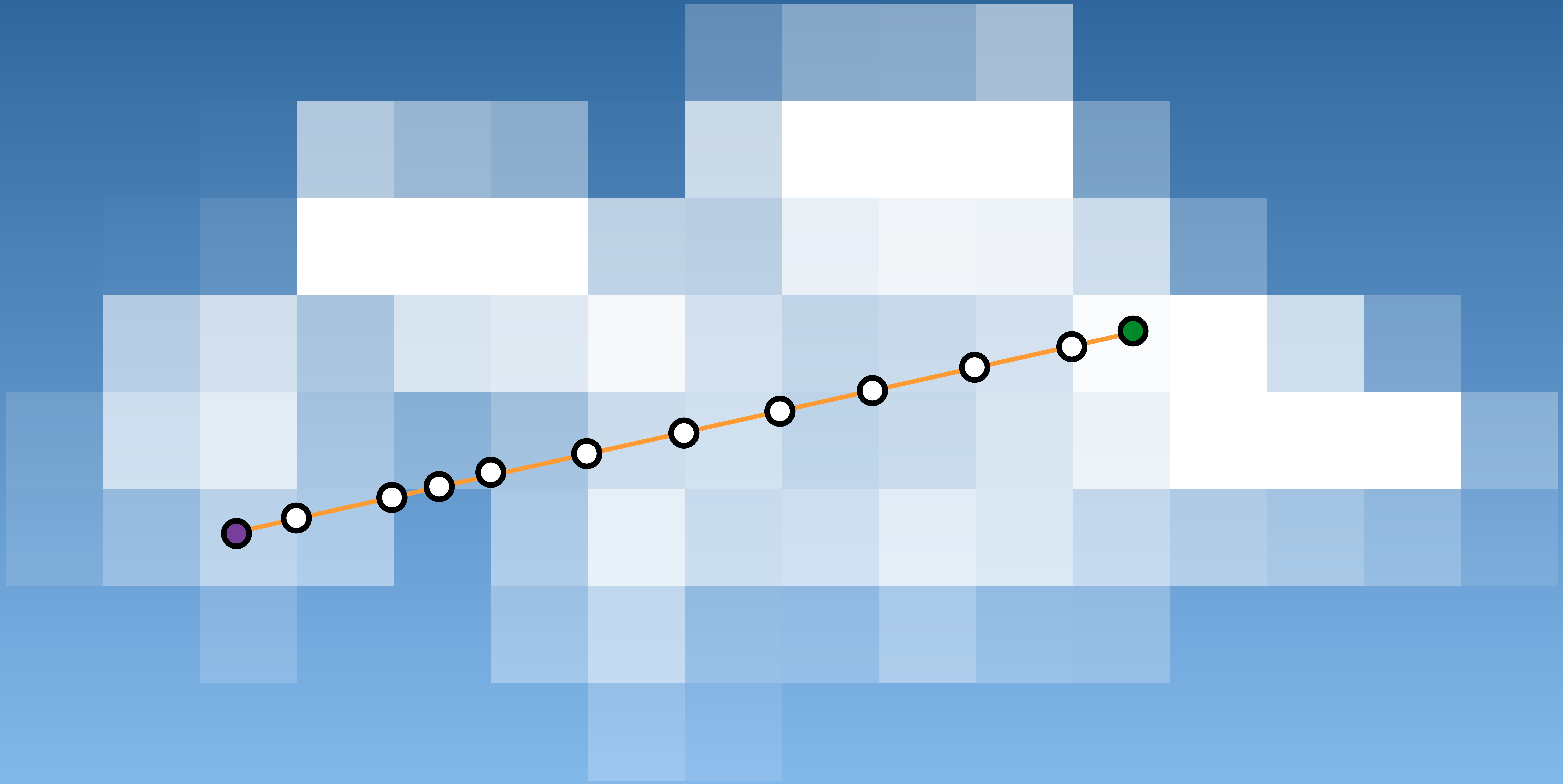

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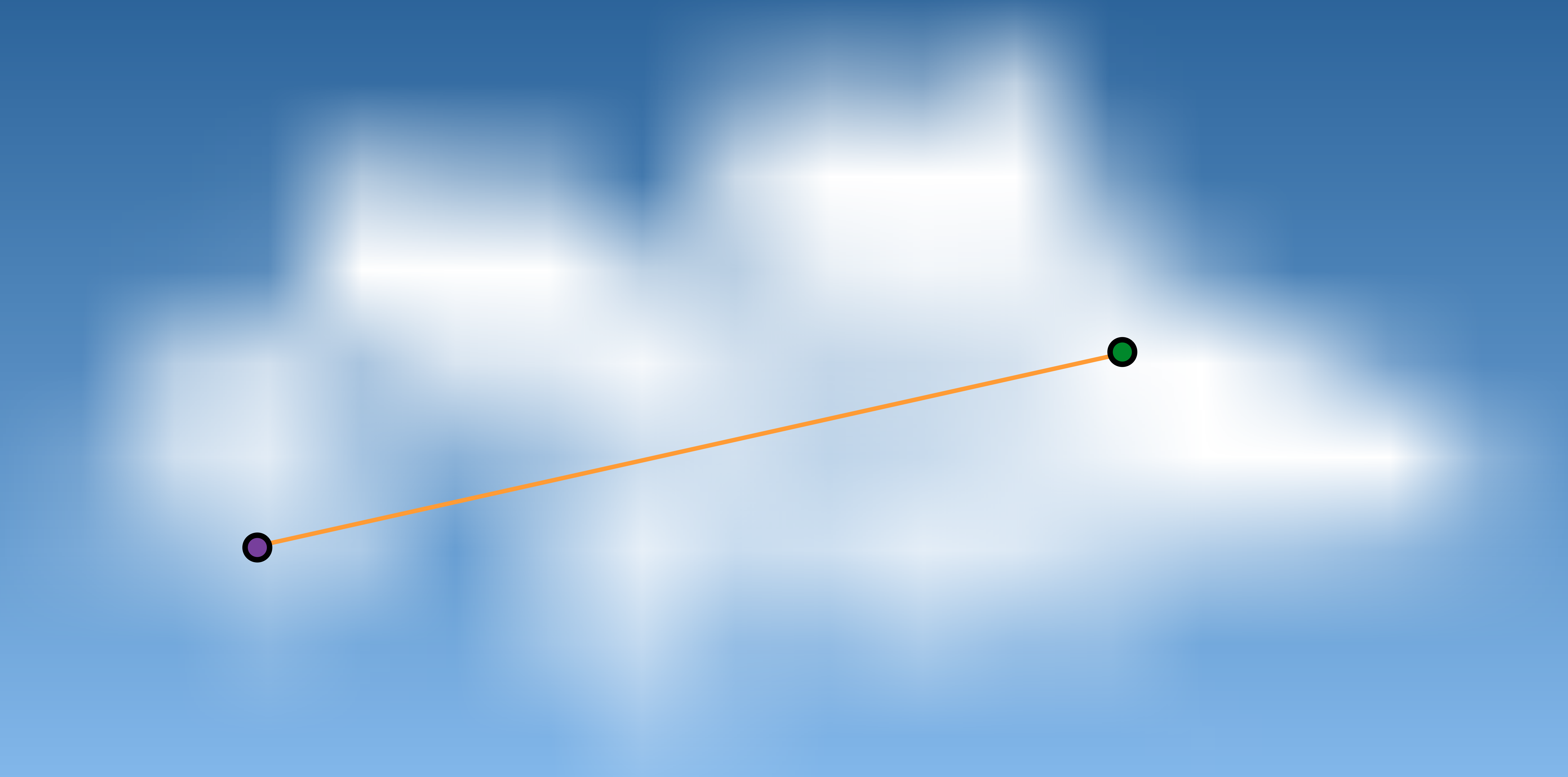
Regular tracking (bi-linear/tri-linear)



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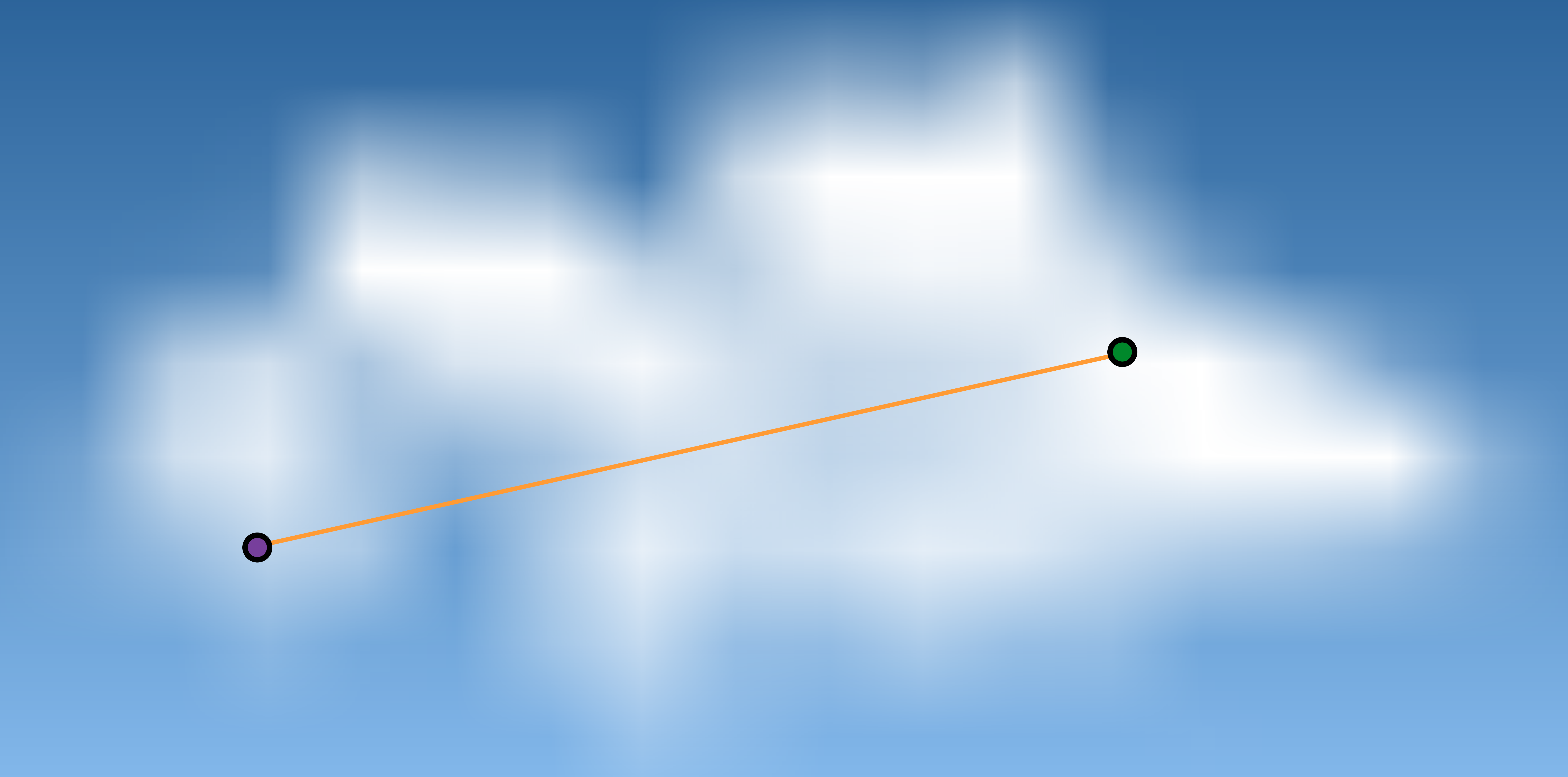

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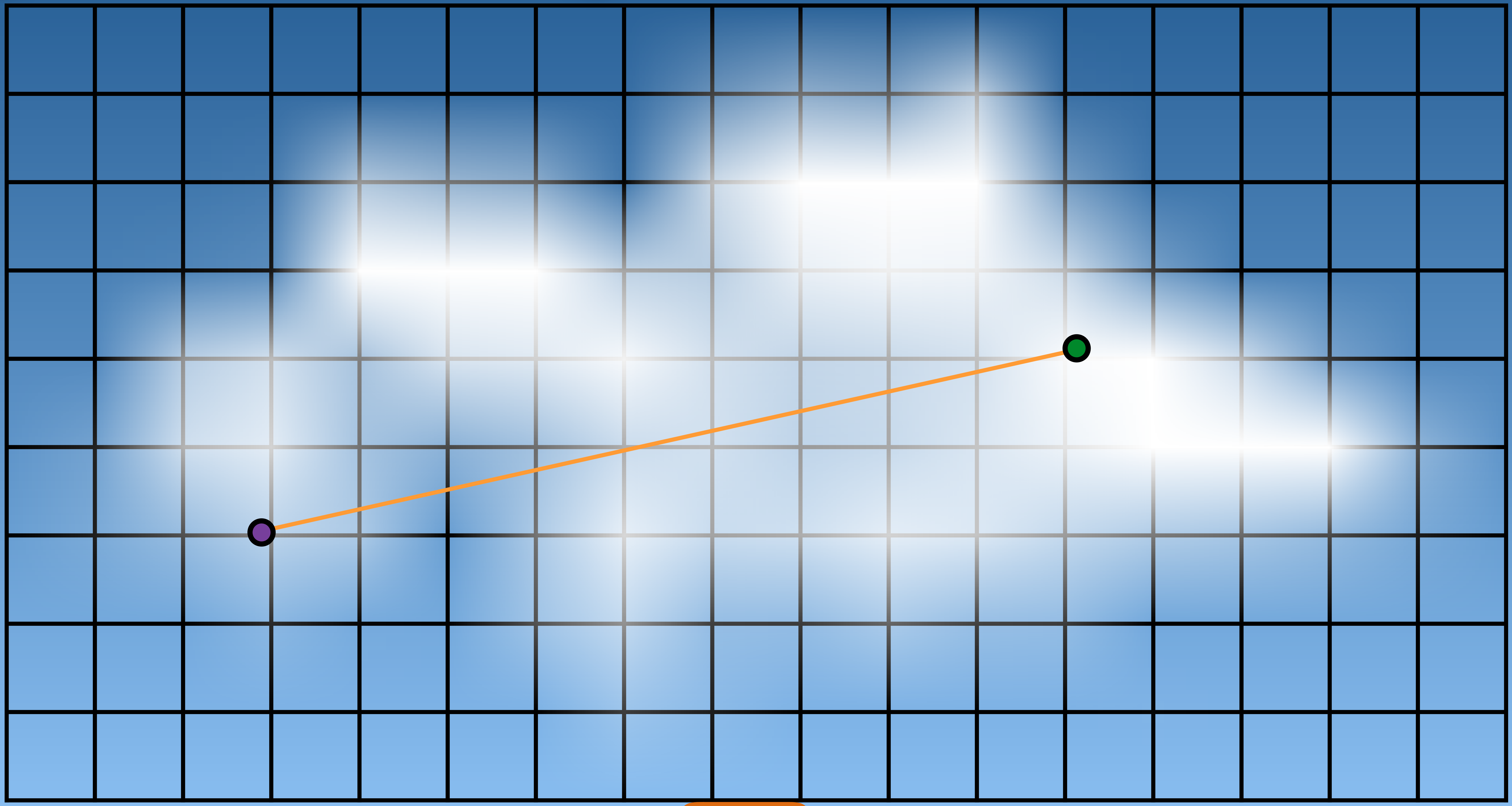

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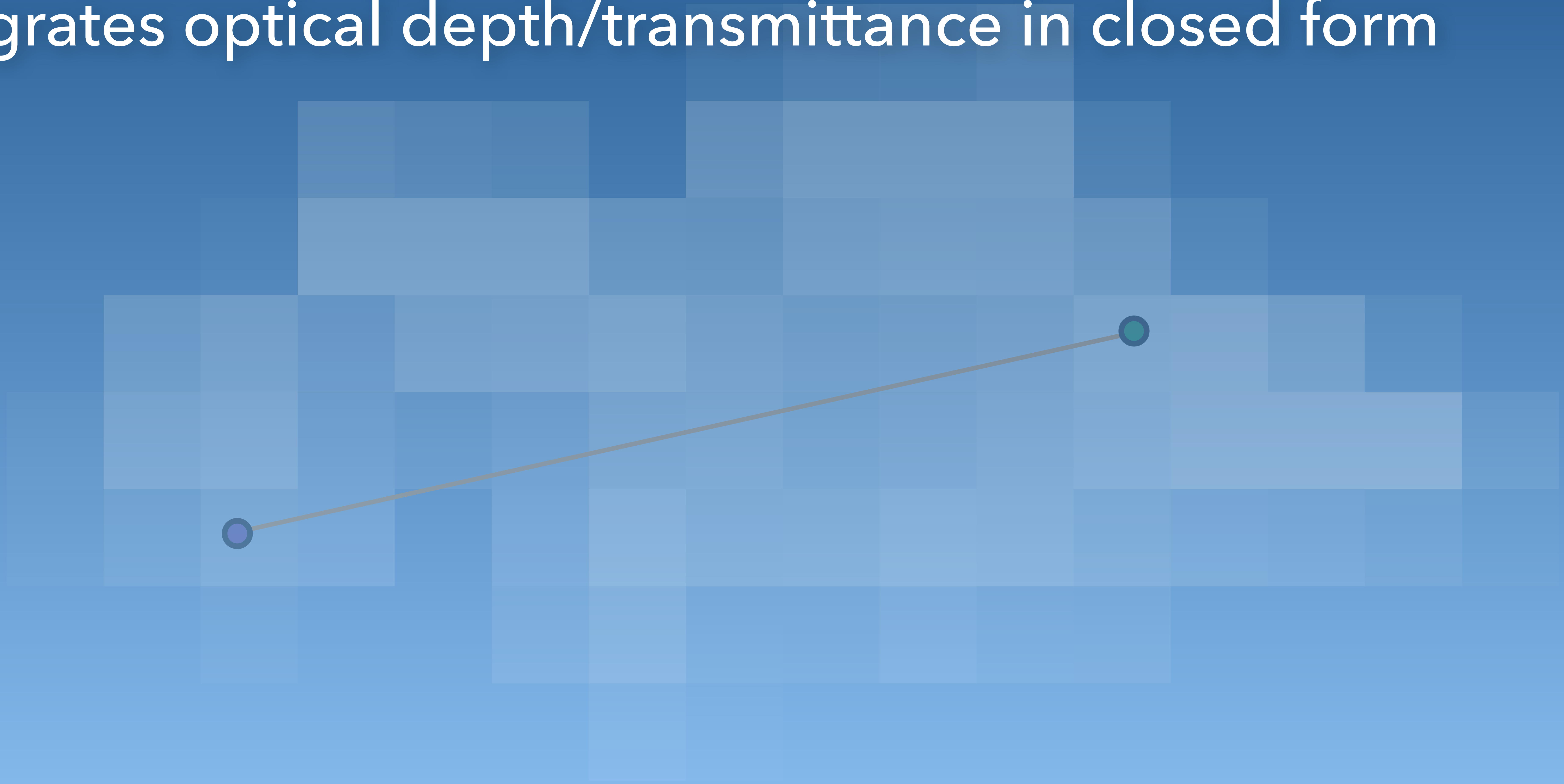
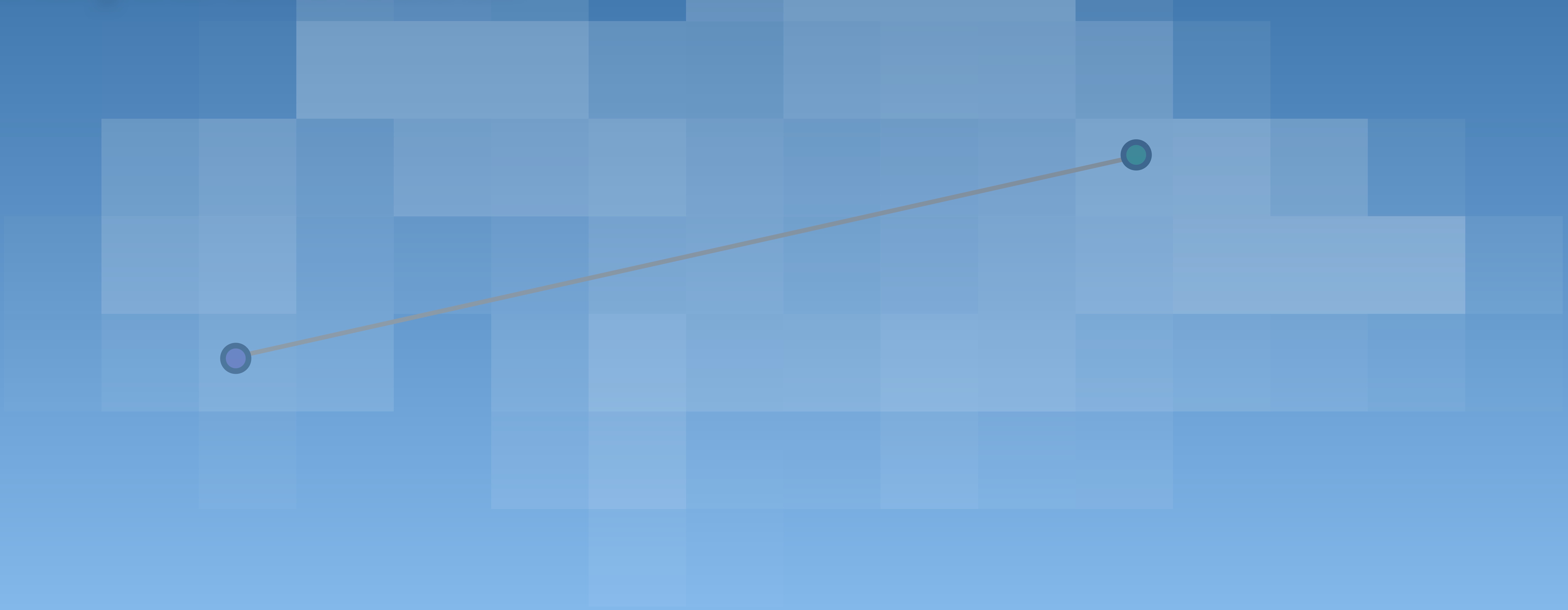
Regular tracking

✔ Integrates optical depth/transmittance in closed form



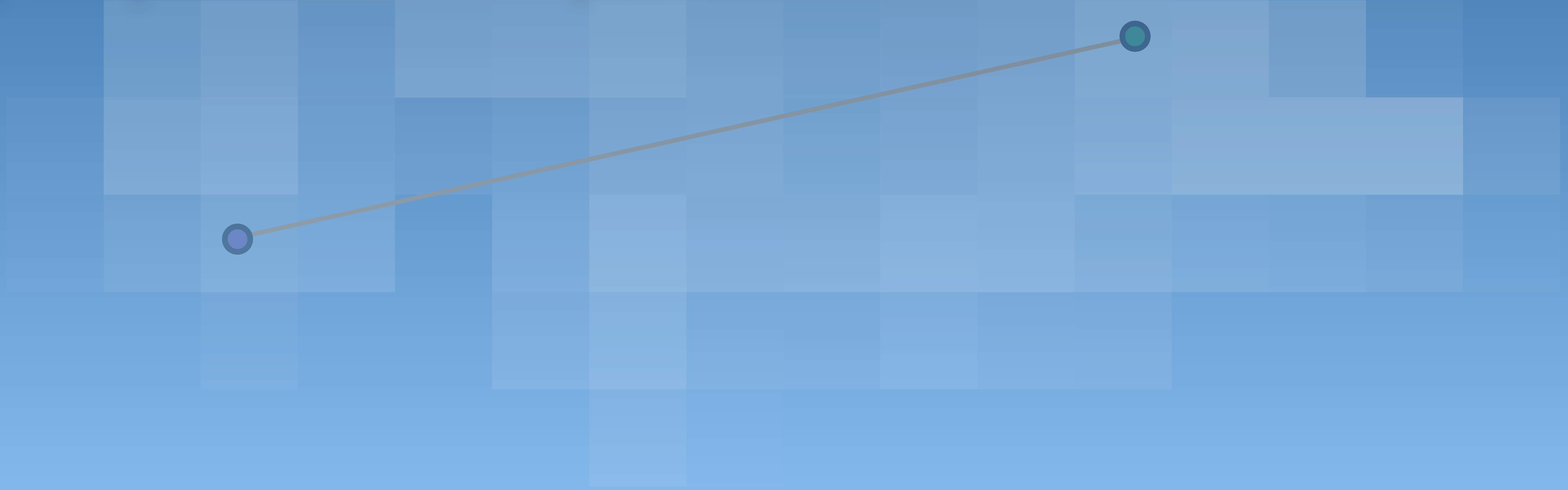
Regular tracking

- ✔ Integrates optical depth/transmittance in closed form
  - ✔ Analytic/unbiased!



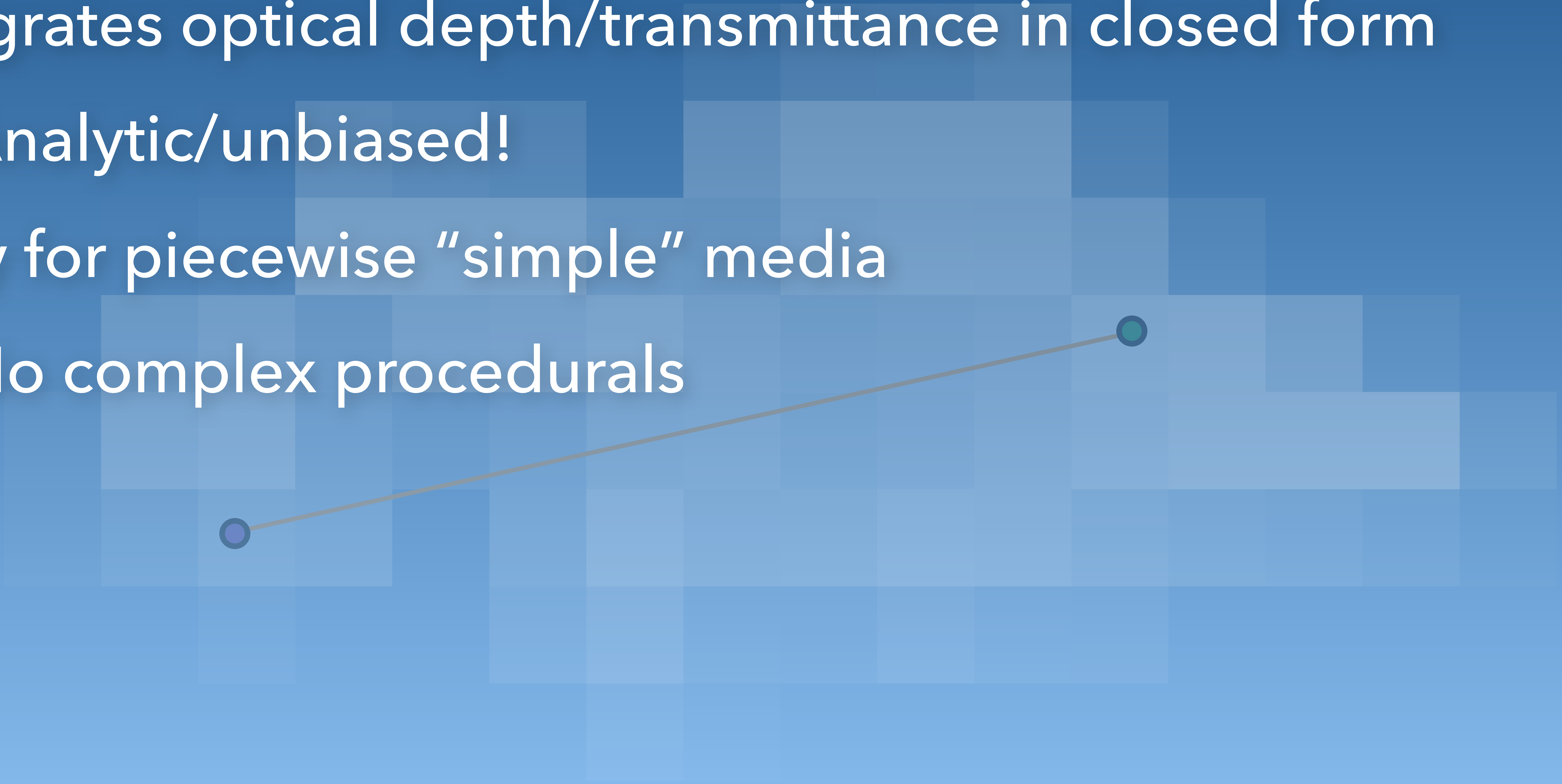
Regular tracking

- ✔ Integrates optical depth/transmittance in closed form
    - ✔ Analytic/unbiased!
- ✘ Only for piecewise “simple” media



Regular tracking

- ✔ Integrates optical depth/transmittance in closed form
  - ✔ Analytic/unbiased!
- ✘ Only for piecewise “simple” media
  - ✘ No complex procedurals



Regular tracking

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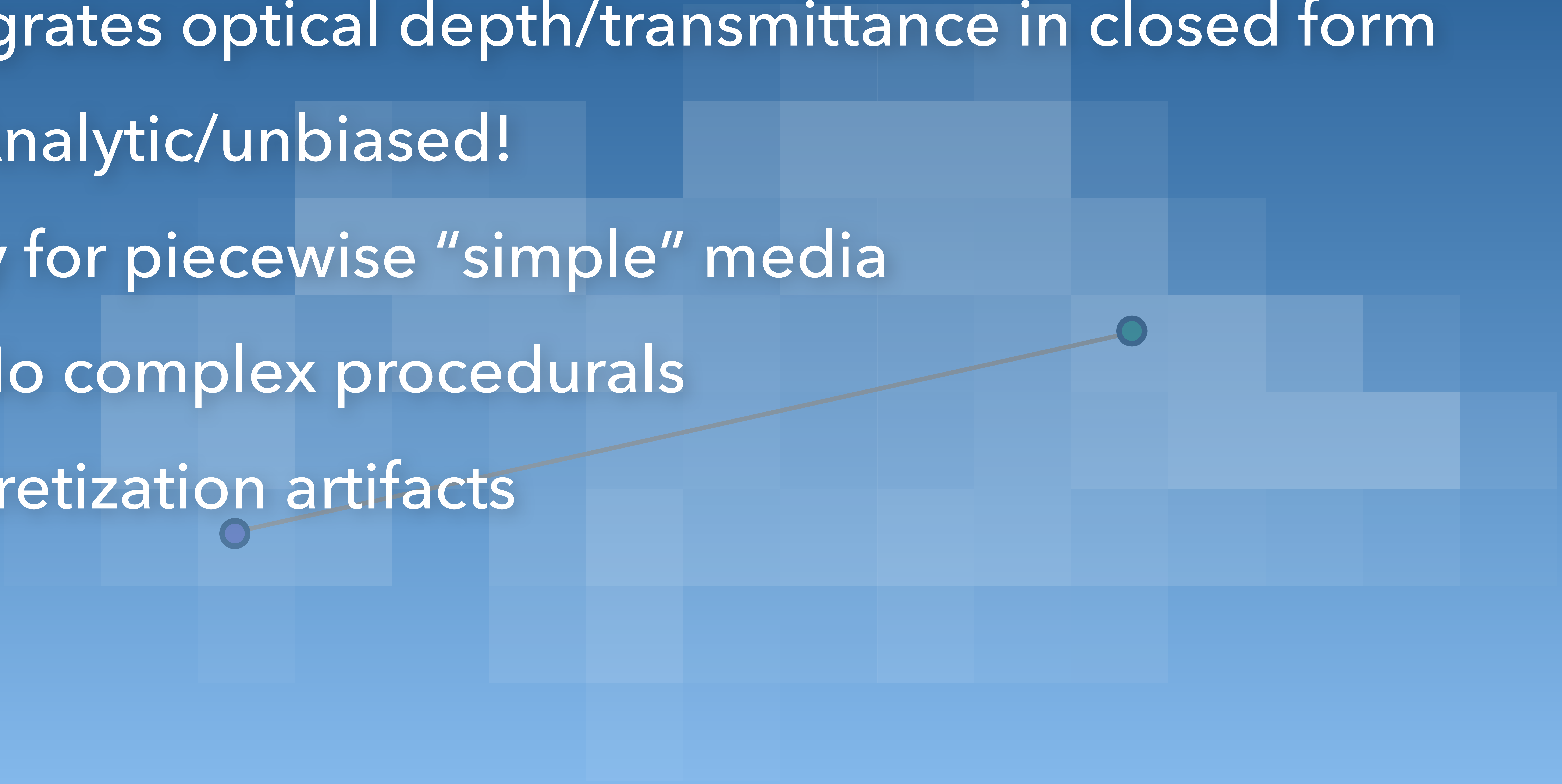
- ✘ Only for piecewise “simple” media
  - ✘ No complex procedurals
- ✘ Discretization artifacts



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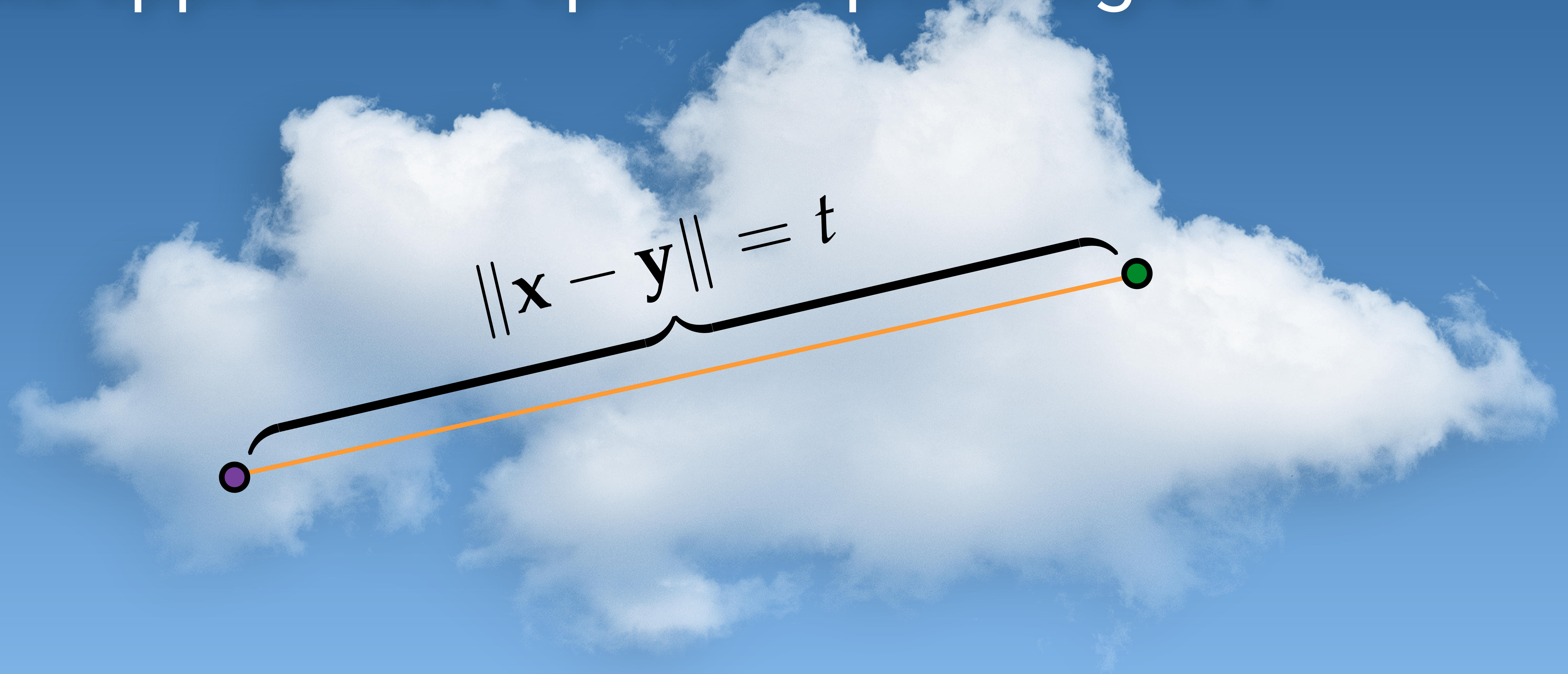
- ✘ Only for piecewise “simple” media
  - ✘ No complex procedurals
- ✘ Discretization artifacts
- ✘ Expensive
- ✘ No speed/quality knob
- ✘ Not progressive

Integrating τ

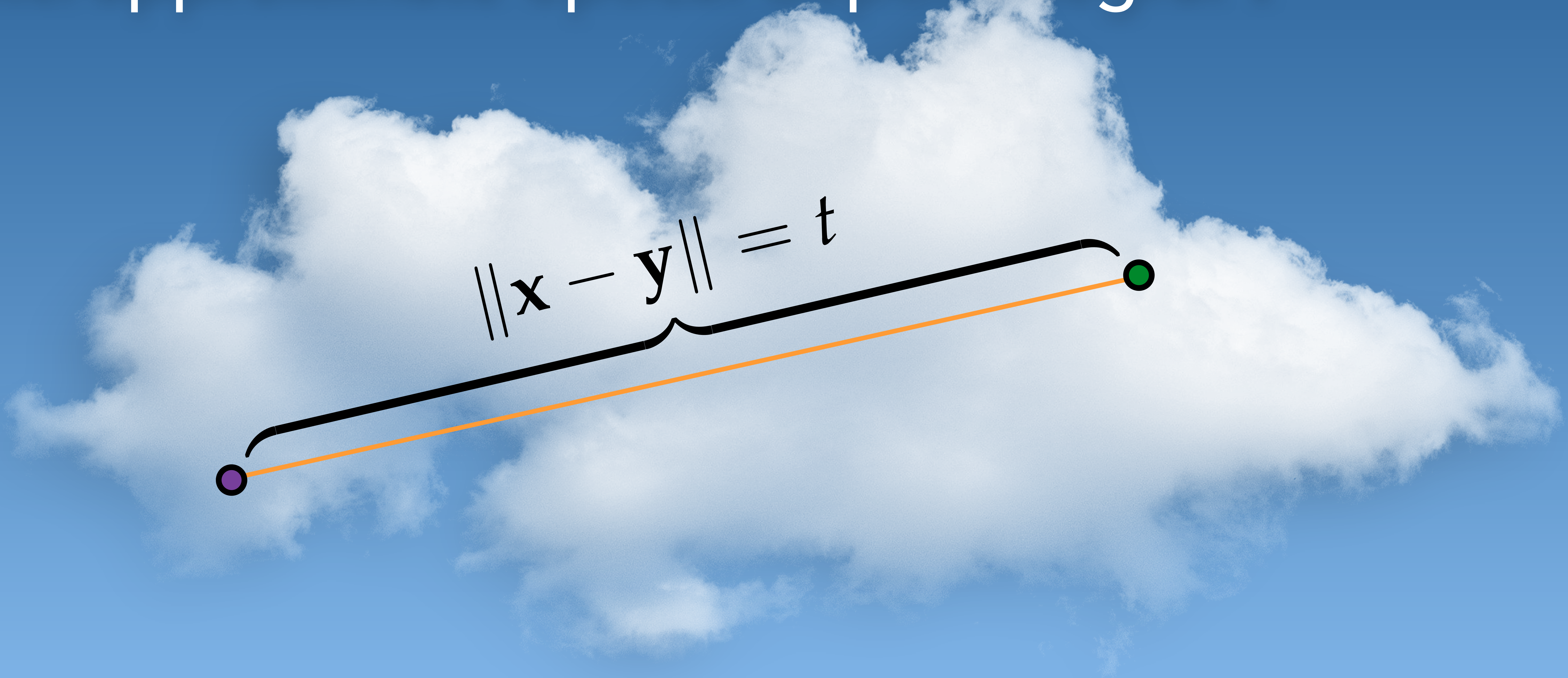
Estimate/approximate optical depth integral τ



$$\tau(t) = \int_0^t \mu_t(t')\,\mathrm{d}t'$$

Integrating τ

Estimate/approximate optical depth integral τ



$$\langle T(t) \rangle_{\mathrm{RM}} = \mathrm{e}^{-\langle \tau(t) \rangle} \qquad \tau(t) = \int_0^t \mu_{\mathrm{t}}(t') \, \mathrm{d}t'$$

Ray marching (Quadrature)

$$\langle T(t) \rangle_{\mathrm{RM}} = \mathrm{e}^{-\langle \tau(t) \rangle}$$

$$\langle \tau(t) \rangle_{\mathrm{RS}} = \sum_{i=1}^{k} \mu_{\mathrm{t}}(t_i)\, \Delta t_i \approx \int_0^t \mu_{\mathrm{t}}(t')\, \mathrm{d}t'$$

Ray marching (Quadrature)

Riemann sum

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Ray marching (Monte Carlo)

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$$\langle \tau(t) \rangle_{\mathrm{MC}} = \sum_{i=1}^{k} \frac{\mu_{\mathrm{t}}(t_i)}{p(t_i)k} \quad \text{with } t_i \propto p(t_i)$$

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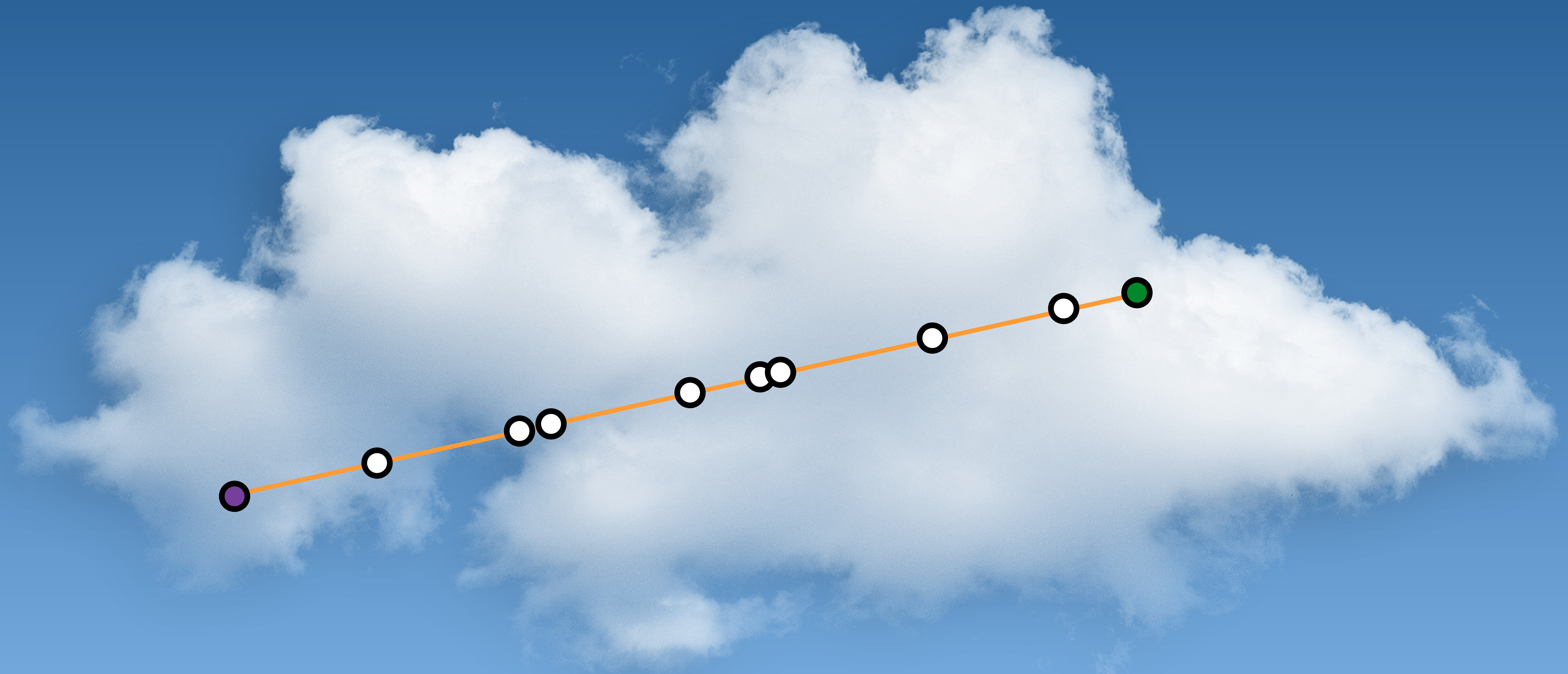

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Ray marching (Stratified Monte Carlo)

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Transmittance from free-flight sampling

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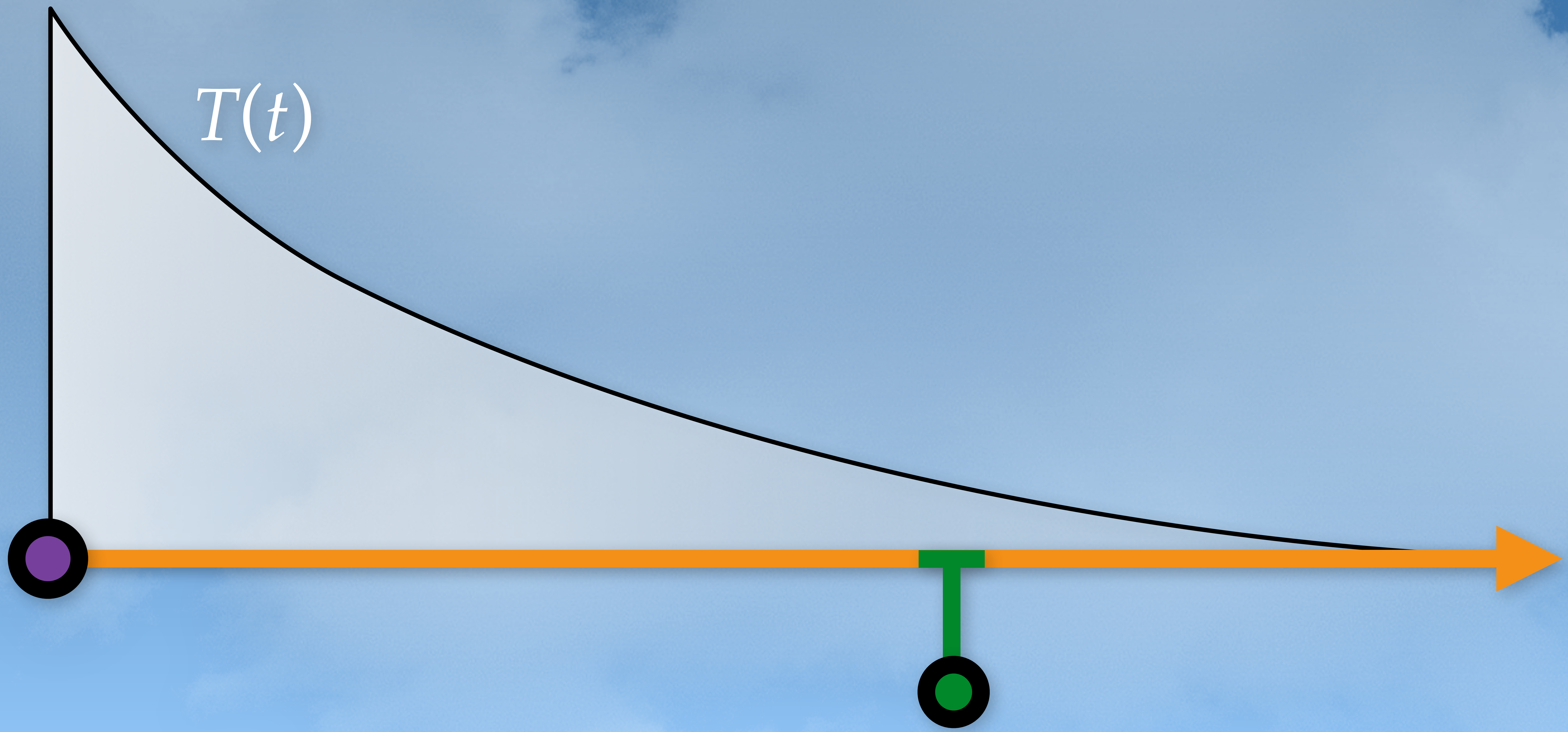

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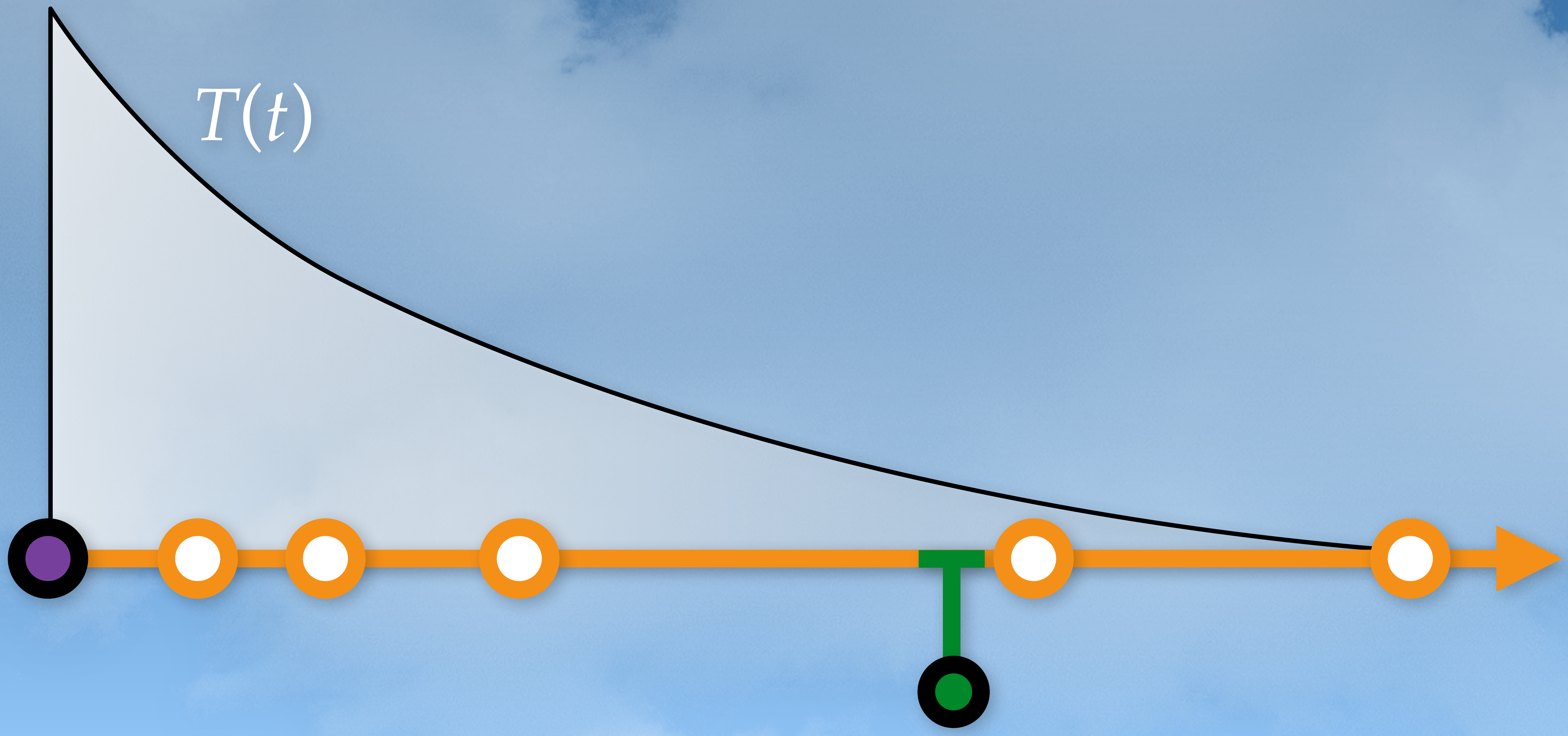
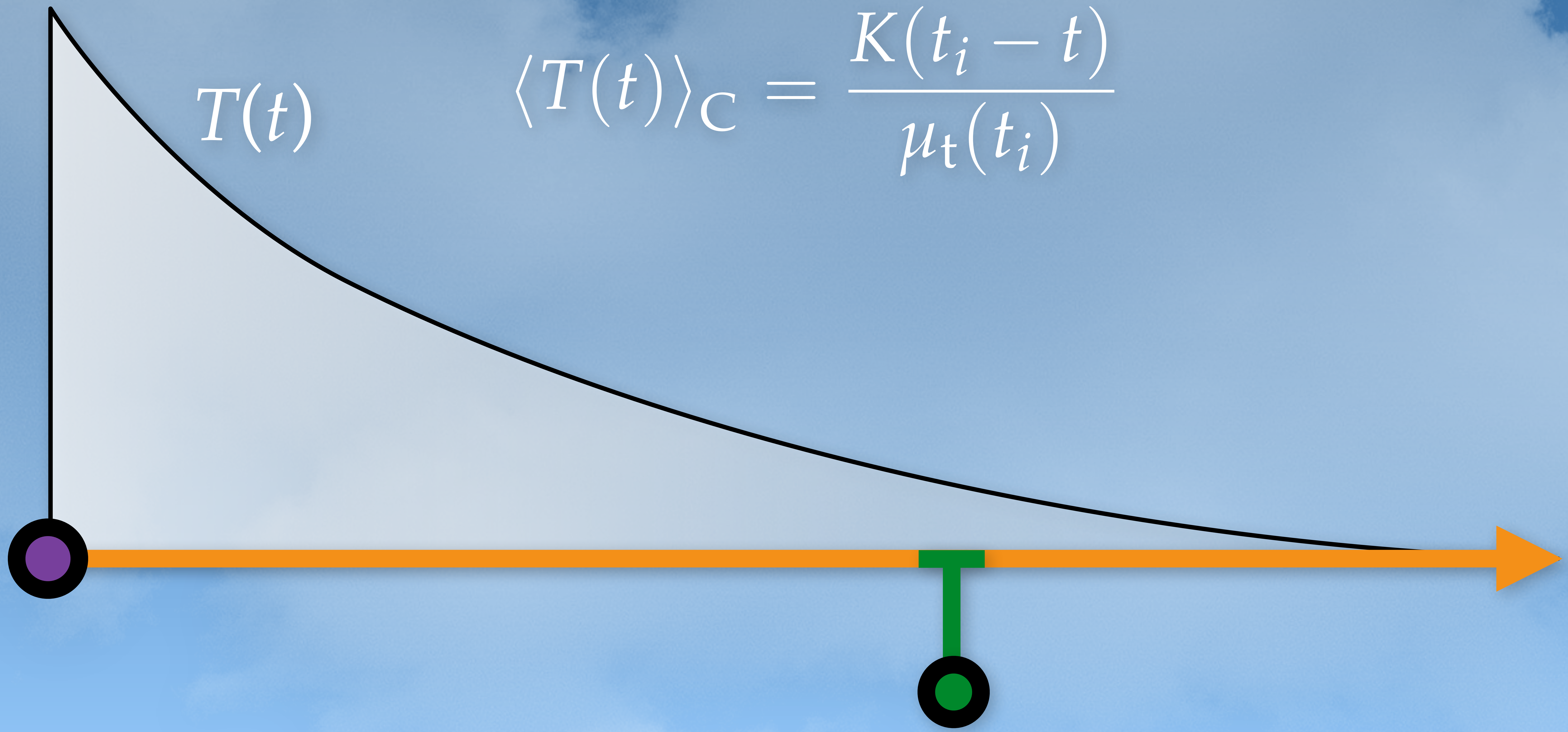


Counting free-flights

Collision estimator:

$$\langle T(t) \rangle_C = \frac{K(t_i - t)}{\mu_t(t_i)}$$



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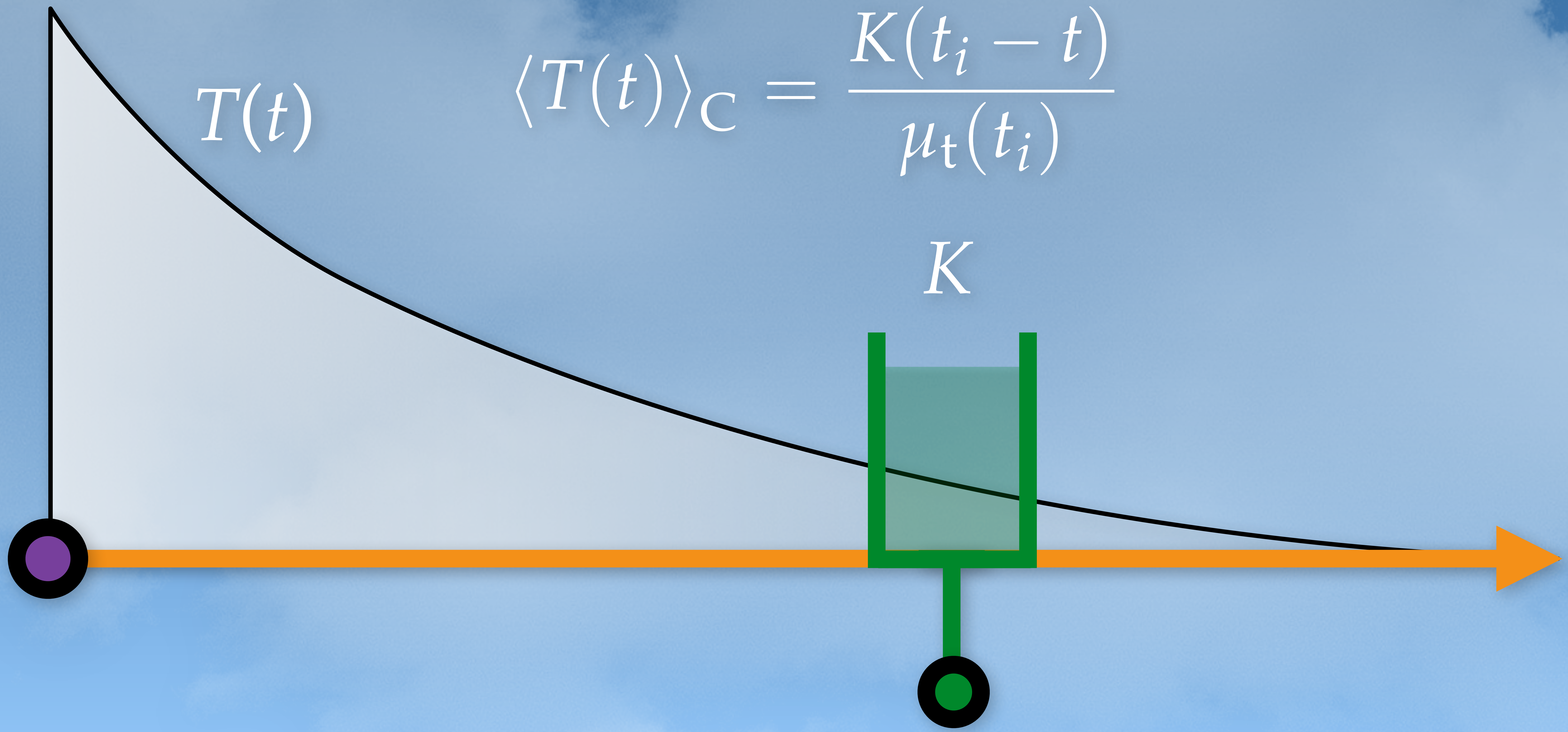

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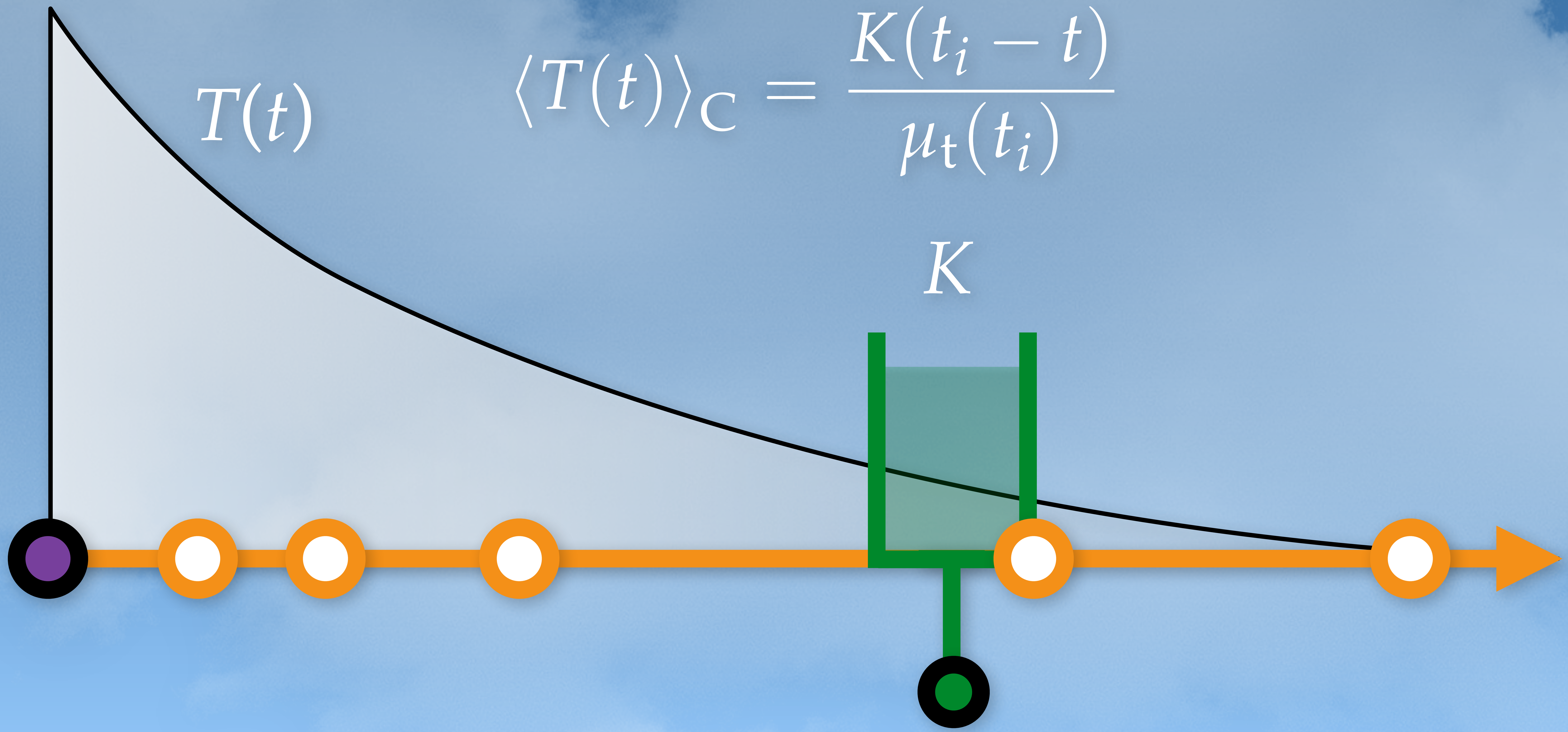

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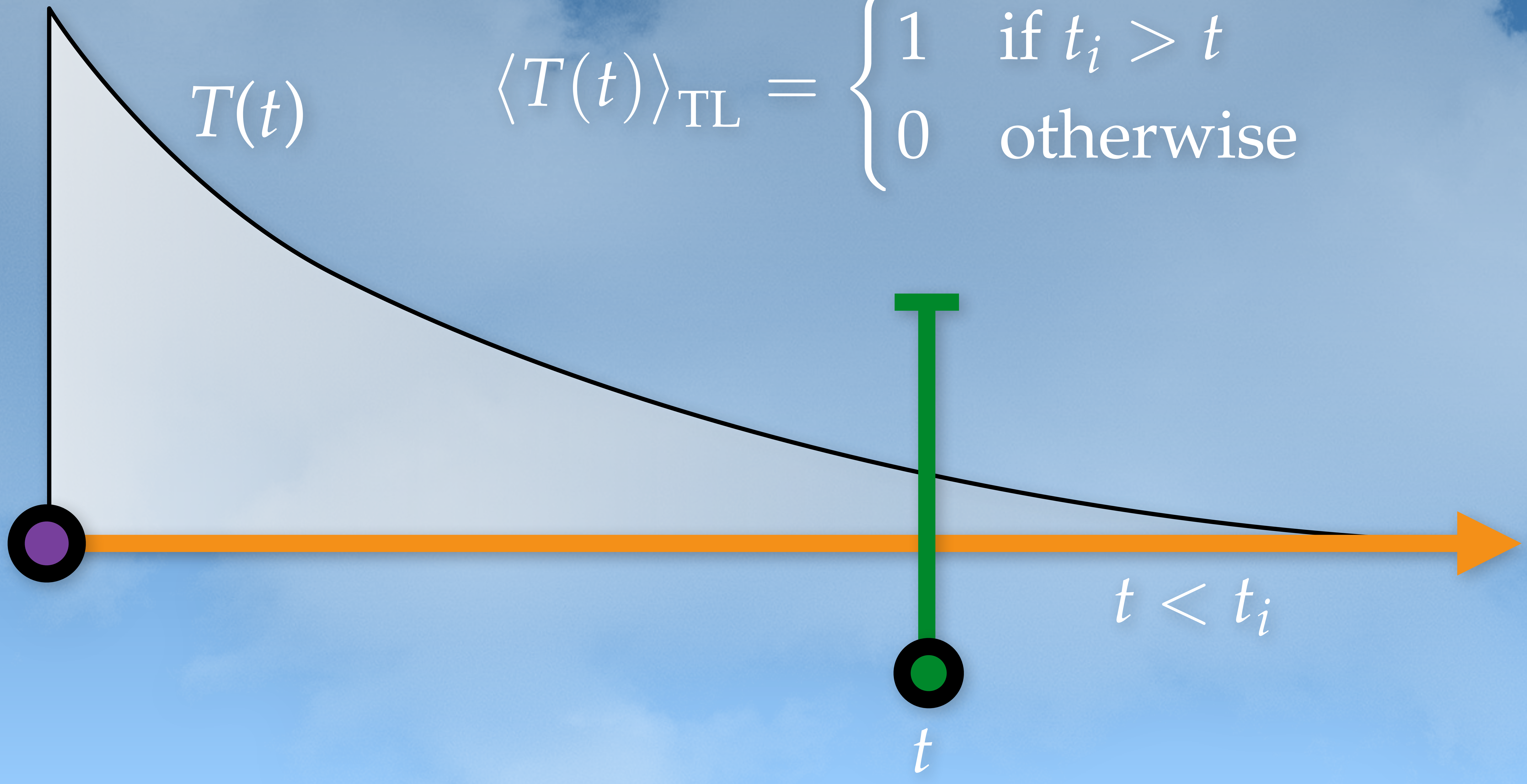
$$\langle T(t) \rangle_{\mathrm{TL}} = \begin{cases} 1 & \text{if } t_i > t \\ 0 & \text{otherwise} \end{cases}$$



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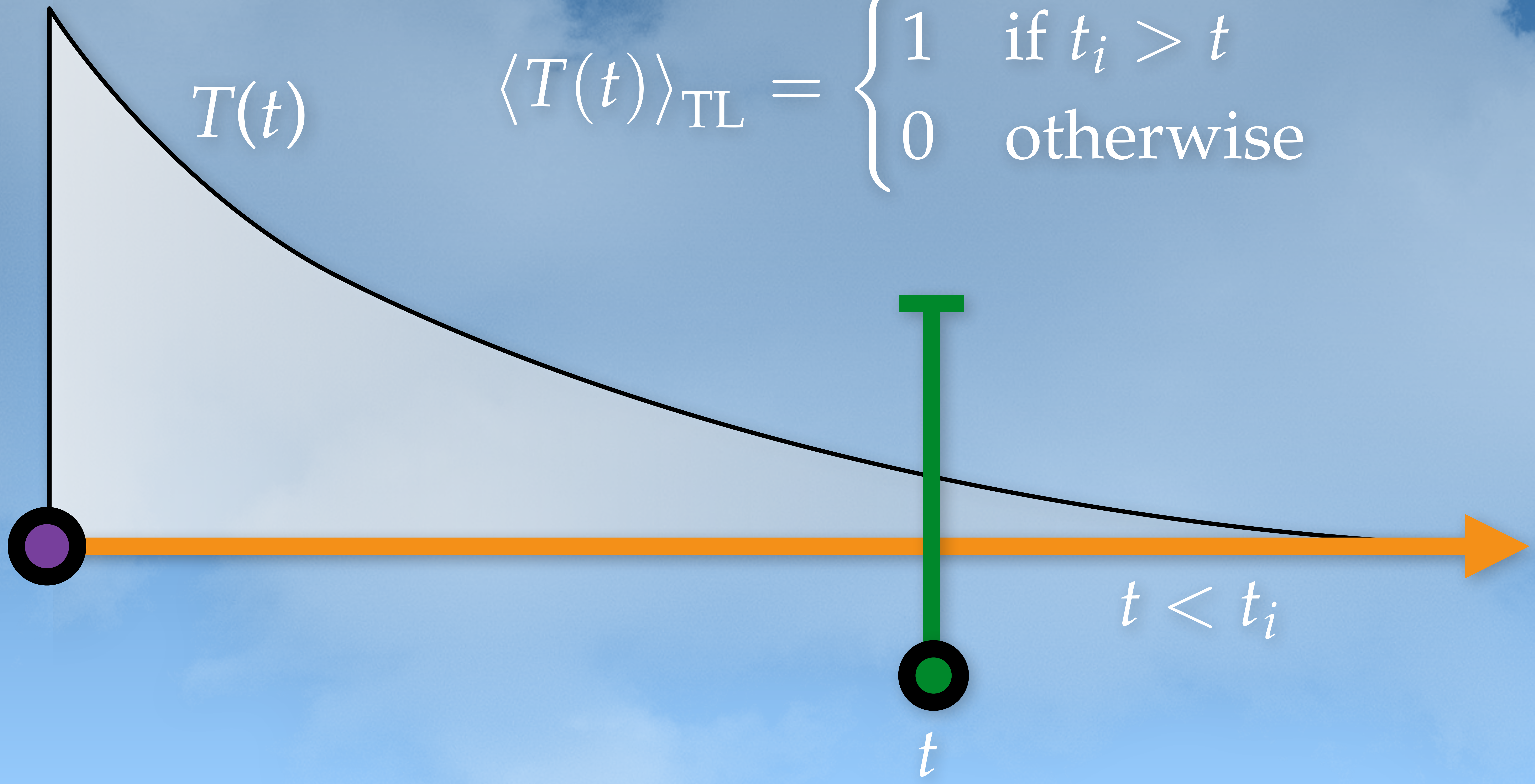

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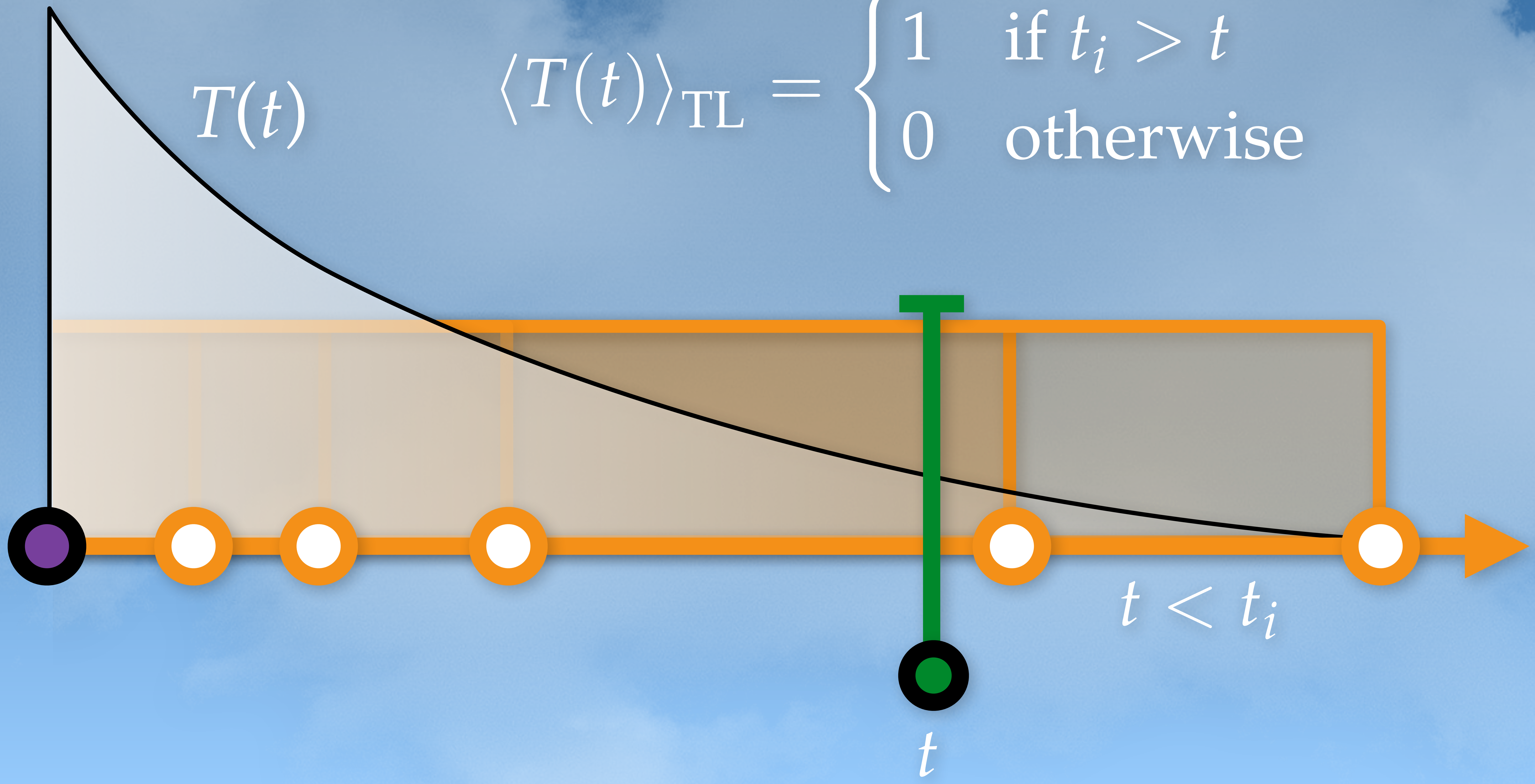

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This gives an **unbiased** "track-length estimator":

$$\langle T(t) \rangle_{\mathrm{TL}} = \begin{cases} \frac{T(t)}{P(t_i > t)} = 1 & \text{if } t_i > t \\ 0 & \text{otherwise} \end{cases}$$

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Transmittance estimation

1. Estimators integrating **optical thickness**
2. Estimators using **free-flight sampling**
3. **Next:** Estimators using **null collisions**

$$T(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{-\tau(\mathbf{x}, \mathbf{y})}$$

transmittance

$$\tau(\mathbf{x}, \mathbf{y}) = \int_0^t \mu_{\mathrm{t}}(\mathbf{x} - s\omega)\,\mathrm{d}s$$

optical thickness

$$p(t_i) \propto T(t_i)$$

free-flight sampling