Quasi-Monte Carlo Quadrature

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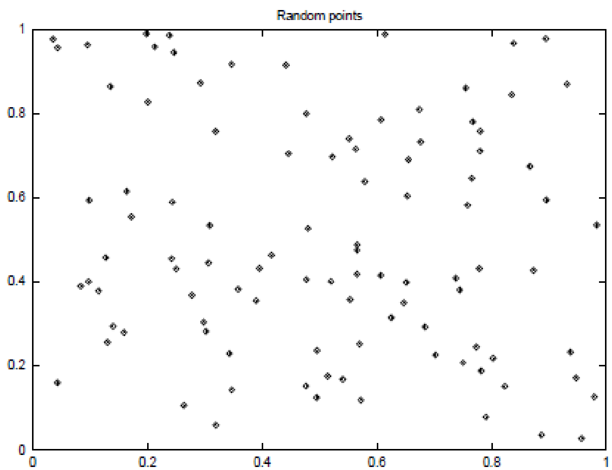
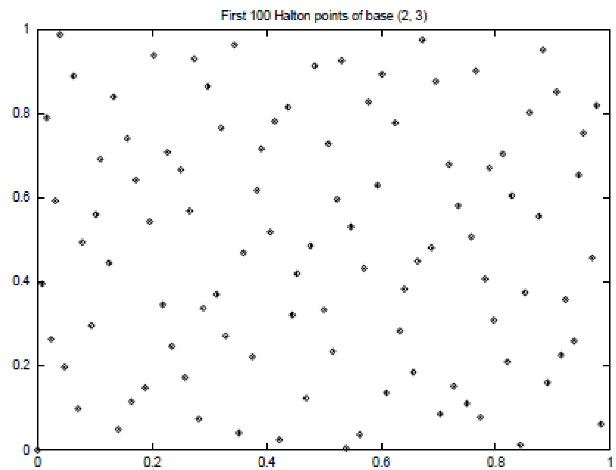
NPGR031

Discrepancy



High Discrepancy
(clusters of points)



Low Discrepancy
(more uniform)

Defining discrepancy

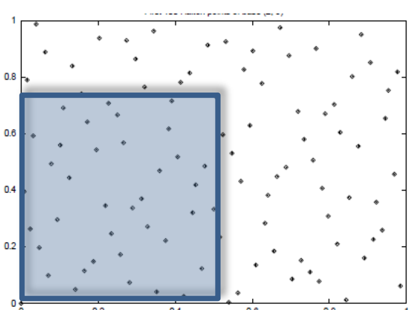
- *s*-dimensional "brick" function:

$$\mathcal{L}(\mathbf{z}) = \begin{cases} 1 \text{ if } 0 \le \mathbf{z}|_1 \le v_1, 0 \le \mathbf{z}|_2 \le v_2, \ldots, 0 \le \mathbf{z}|_s \le v_s \\ 0 \text{ otherwise.} \end{cases}$$

- True volume of the "brick" function:

$$V(A) = \prod_{j=1}^{s} v_j$$

- MC estimate of the volume of the "brick":



$$\frac{1}{N}\sum_{i=1}^{N} f(\mathbf{z}_i) = \frac{m(A)}{N}$$

total number of sample points

number of sample points that actually fell inside the "brick"

Discrepancy

- Discrepancy (of a point sequence) is the maximum possible error of the MC quadrature of the “brick” function over all possible brick shapes:

$$\mathcal{D}^*(\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_N) = \sup_A \left|\frac{m(A)}{N} - V(A)\right|.$$

  - serves as a measure of the uniformity of a point set
  - must converge to zero as N -> infty
  - the lower the better (cf. **Koksma-Hlawka Inequality**)

Koksma-Hlawka inequality

- Hardy-Krause Variation

$$\mathcal{V}_{\mathrm{HK}}(f(u,v)) = \int_0^1 \int_0^1 \left| \frac{\partial^2 f(u,v)}{\partial u \partial v} \right| du\, dv + \int_0^1 \left| \frac{\partial f(u,1)}{\partial u} \right| du + \int_0^1 \left| \frac{\partial f(1,v)}{\partial v} \right| dv$$

- Koksma-Hlawka inequality

$$\left| \int_{\mathbf{z} \in [0,1]^s} f(\mathbf{z})\, d\mathbf{z} - \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{z}_i) \right| \leq \mathcal{V}_{\mathrm{HK}} \cdot \mathcal{D}^*(\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_N)$$

  – the KH inequality only applies to *f* with finite variation
  – QMC can still be applied even if the variation of f is infinite

Low-discrepancy Sequences

- Lloyd relaxation
- Poison disk distribution
- Van der Corput Sequence
- Halton / Hammersley sequence

Radical Inversion-based sequences

- Van der Corput Sequence

$$\begin{aligned} \Phi_b : \mathbb{N}_0 &\rightarrow \mathbb{Q} \cap [0,1) \\ i = \sum_{j=0}^{\infty} a_j(i) b^j &\mapsto \Phi_b(i) := \sum_{j=0}^{\infty} a_j(i) b^{-j-1} \end{aligned}$$

```
double RadicalInverse(const int Base, int i)
{
        double Digit, Radical, Inverse;
        Digit = Radical = 1.0 / (double) Base;
        Inverse = 0.0;
        while(i)
        {
                Inverse += Digit * (double) (i % Base);
                Digit *= Radical;
                i /= Base;
        }
        return Inverse;
}
```

Van der Corput Sequence (base 2)

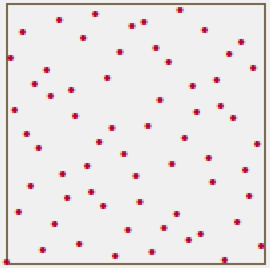
i	binary form of i	radical inverse	H_i
1	1	0.1	0.5
2	10	0.01	0.25
3	11	0.11	0.75
4	100	0.001	0.125
5	101	0.101	0.625
6	110	0.011	0.375
7	111	0.111	0.875

- point placed in the middle of the interval
- then the interval is divided in half

Table credit: Laszlo Szirmay-Kalos

Radical inversion based points in higher dimension

Halton sequence $x_i := (\Phi_{b_1}(i), \ldots, \Phi_{b_s}(i))$ where b_i is the i-th prime number



Hammersley point set $x_i := \left(\frac{i}{n}, \Phi_{b_1}(i), \ldots, \Phi_{b_{s-1}}(i)\right)$

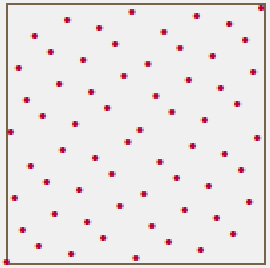


Image credit: Alexander Keller

MC vs. QMC

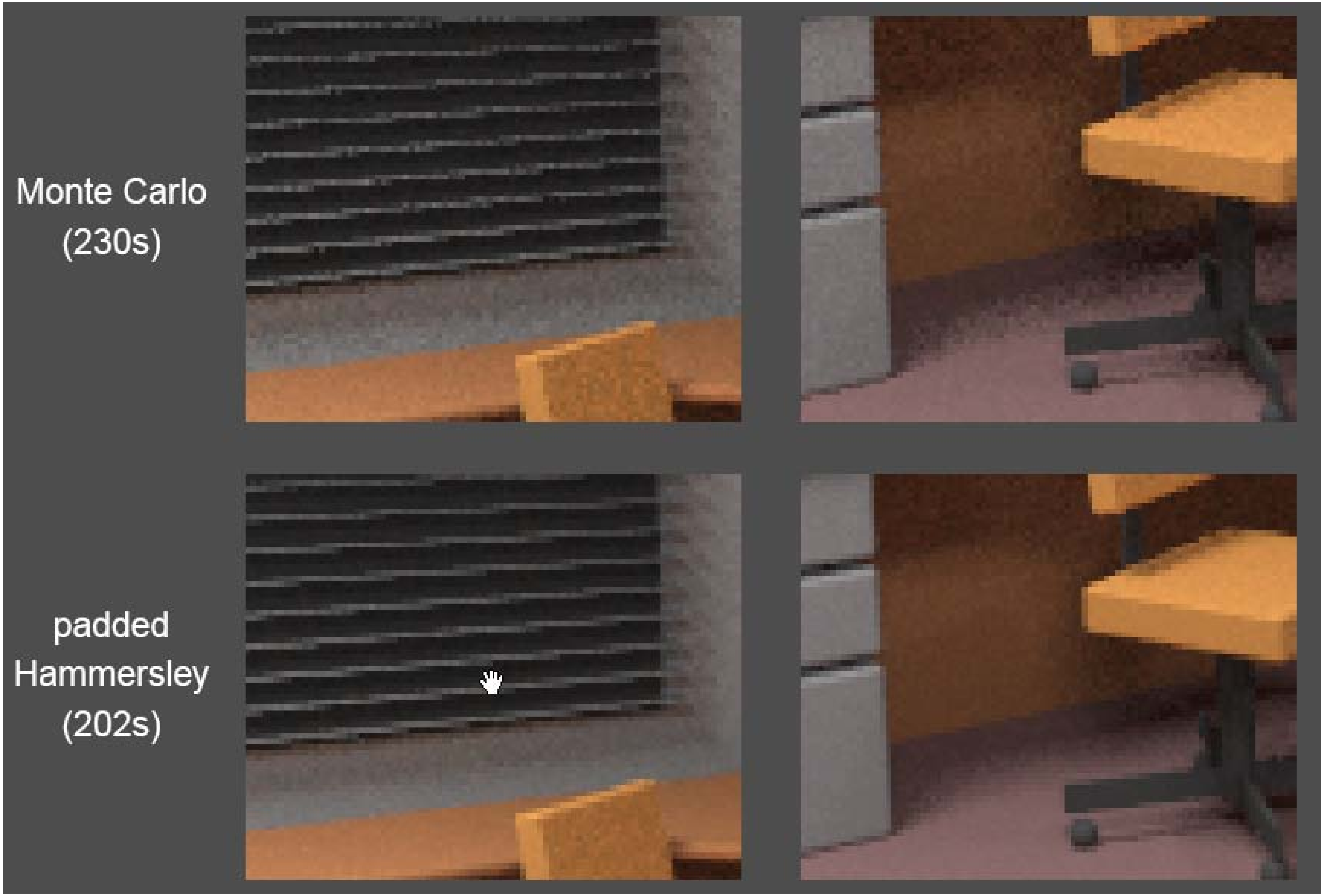
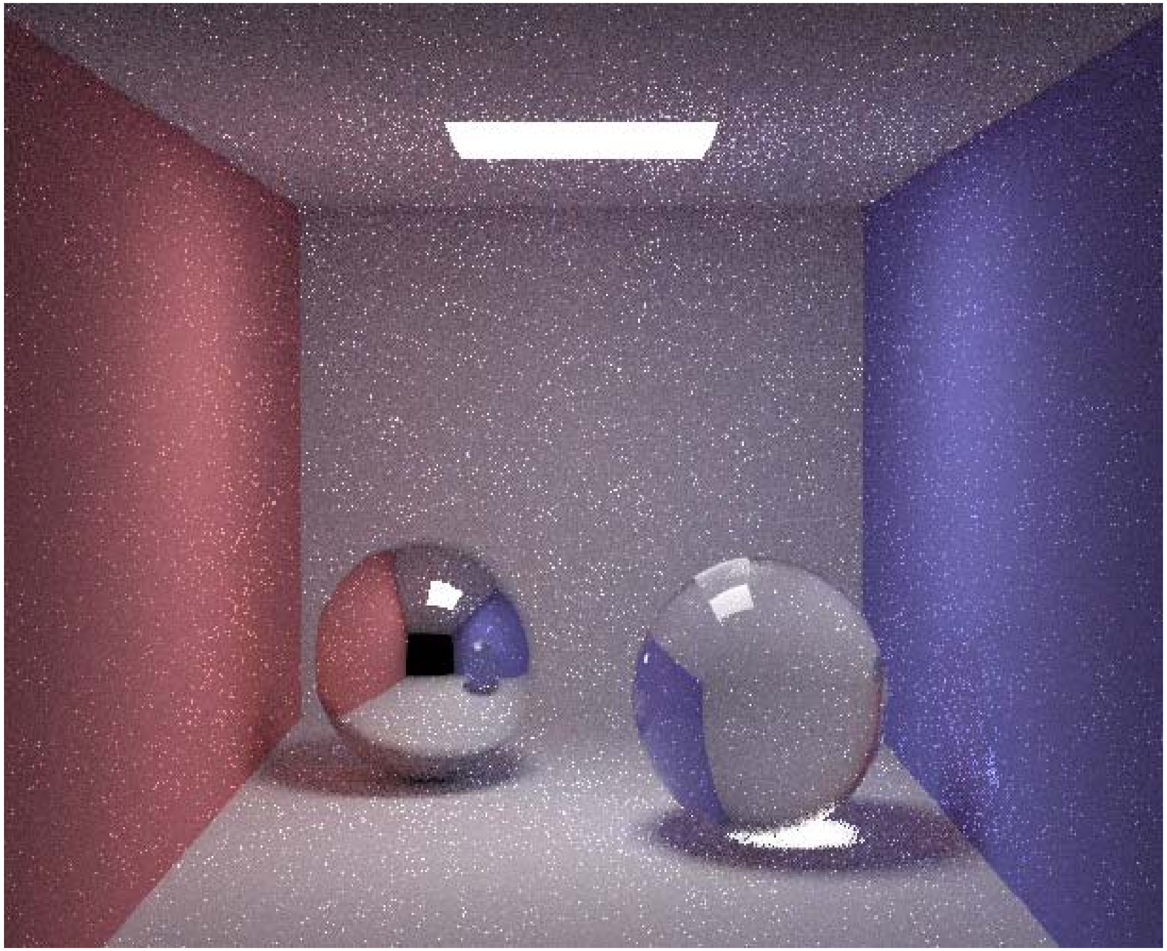


Image credit: Alexander Keller

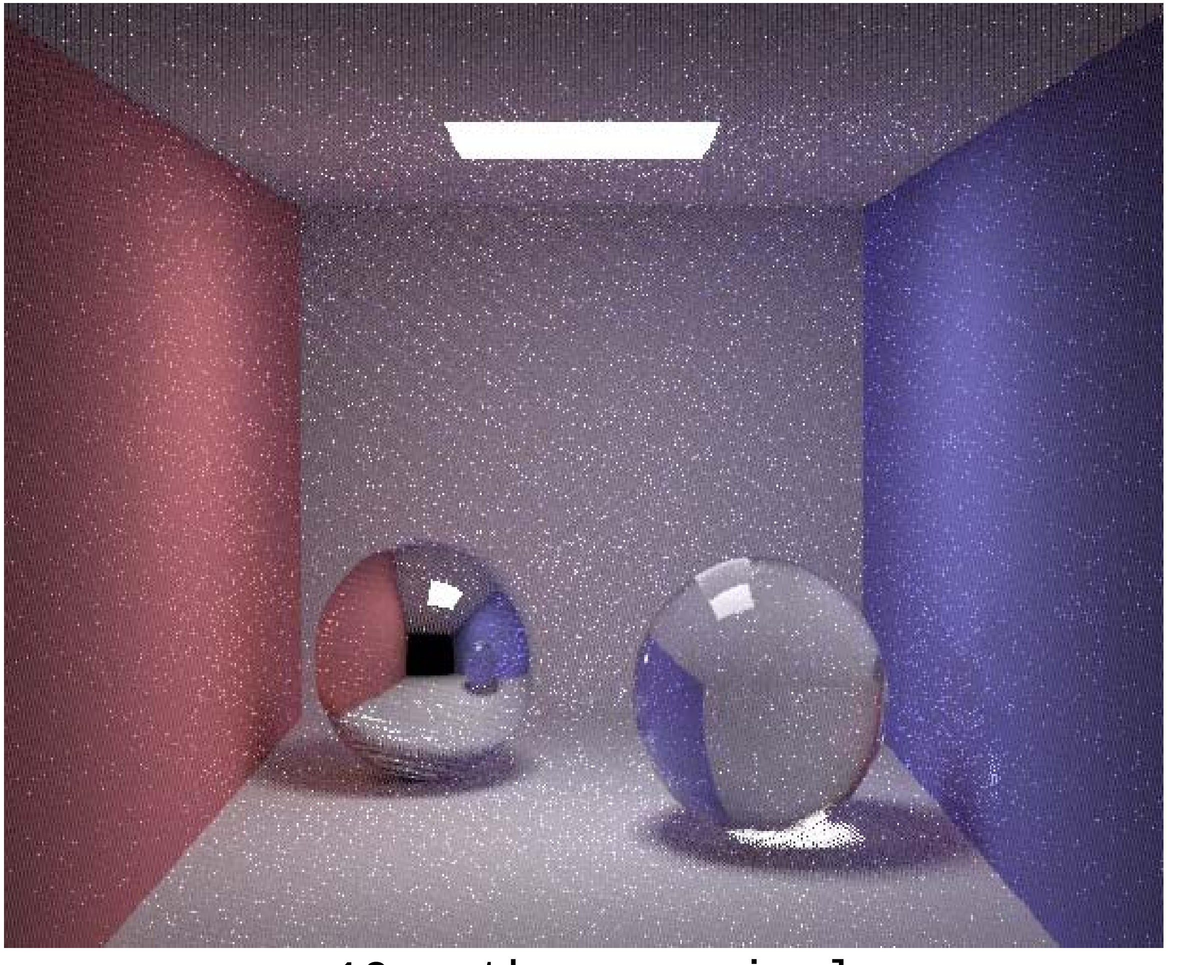
Stratified sampling



10 paths per pixel

Image credit: Henrik Wann Jensen

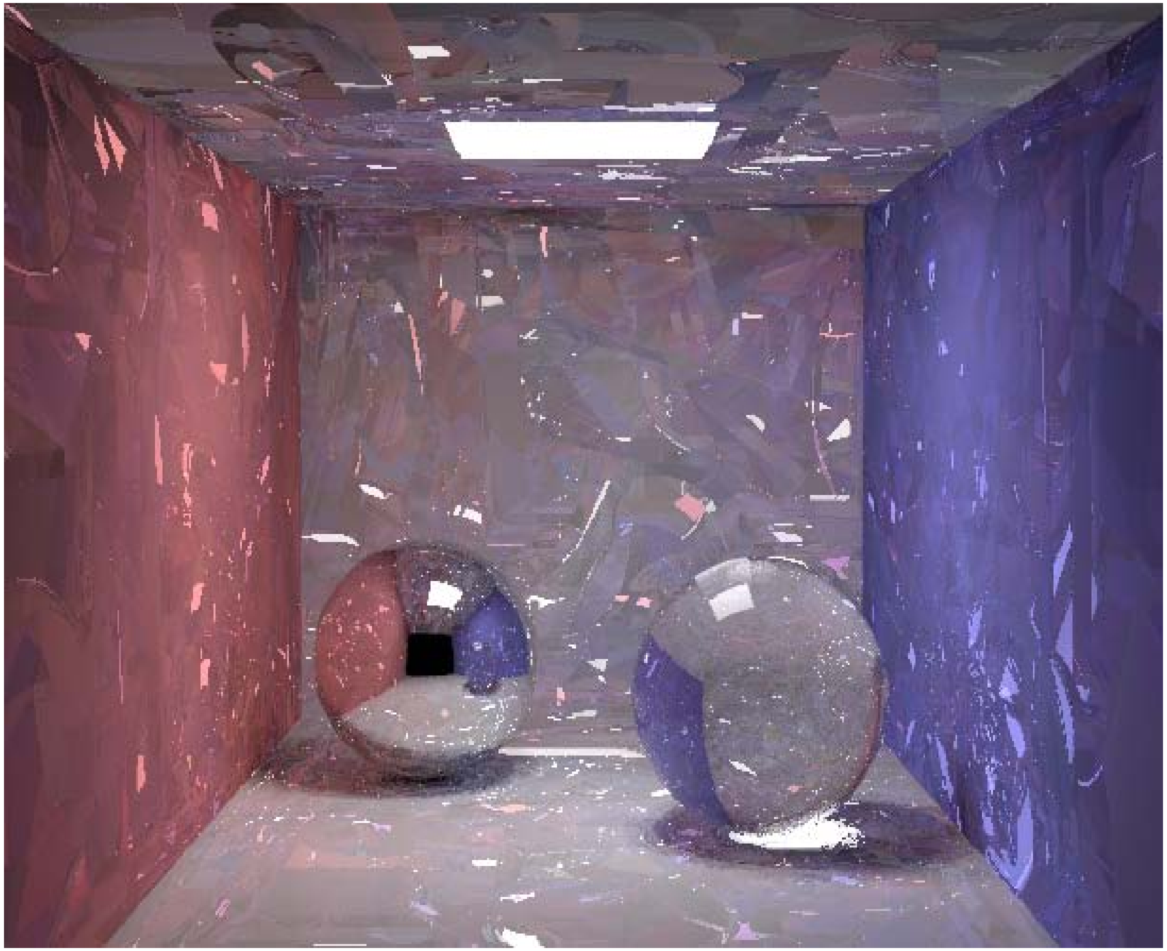
Quasi-Monte Carlo



10 paths per pixel

Image credit: Henrik Wann Jensen

Fixed sequence



10 paths per pixel

Image credit: Henrik Wann Jensen

References

- Szirmay-Kalos: Monte Carlo methods in global Illumination, pp. 42 – 48.
- Pharr and Humphreys: PBRT, chapter 7 (Sampling and Reconstruction)