Function Representation & Spherical Harmonics

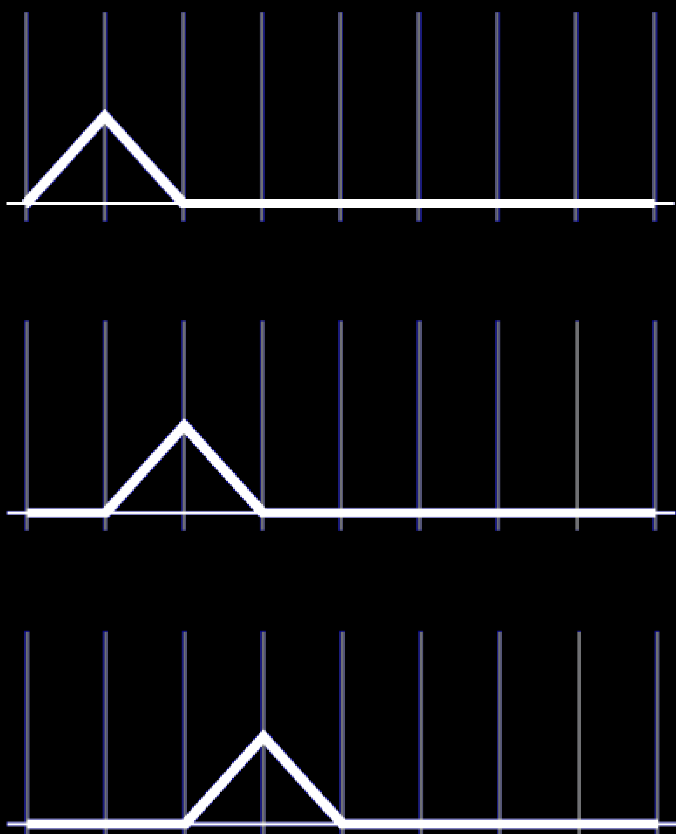
Function approximation

- $G(x)$ ... function to represent
- $B_1(x)$, $B_2(x)$, … $B_n(x)$ … basis functions
- $G(x)$ is a linear combination of basis functions

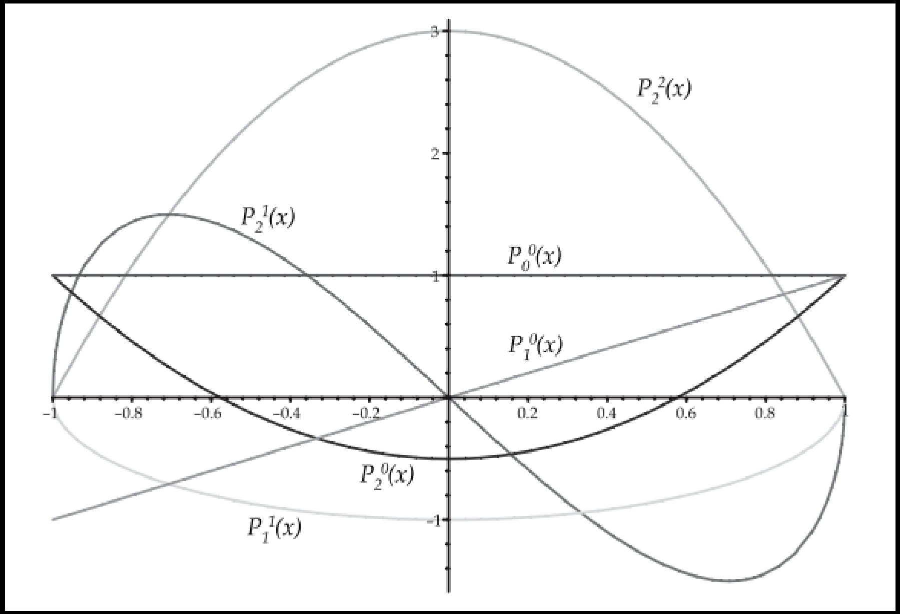
$$G(x) = \sum_{i=1}^{n} c_i B_i(x)$$

- Storing a finite number of coefficients c_i gives an approximation of $G(x)$

Examples of basis functions



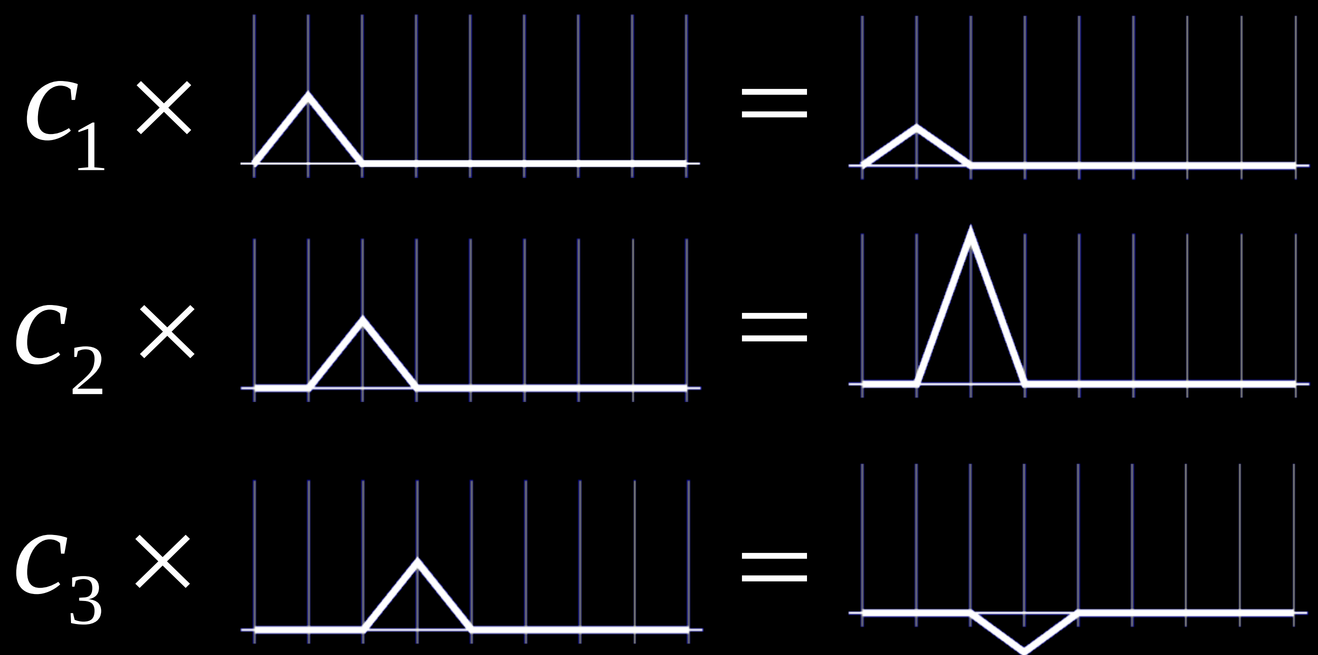
Tent function (linear interpolation)



Associated Legendre polynomials

Function approximation

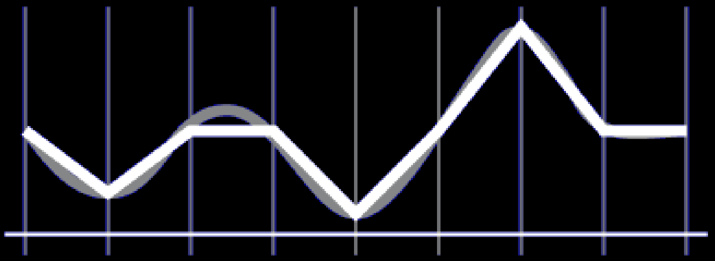
- Linear combination
  - sum of scaled basis functions



Function approximation

- Linear combination
  - sum of scaled basis functions

$$\sum_{i=1}^{n} c_i B_i(x) =$$



Finding the coefficients

- How to find coefficients c_i?
  - Minimize an error measure
- What error measure?
  - L_2 error

$$E_{L_2} = \int_I [\underbrace{G(x)}_{\text{Original function}} - \underbrace{\sum_i c_i B_i(x)}_{\text{Approximated function}}]^2$$

Original function

Approximated function

Finding the coefficients

- Minimizing E_{L_2} leads to

$$\begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle G | B_1 \rangle \\ \langle G | B_2 \rangle \\ \vdots \\ \langle G | B_n \rangle \end{bmatrix}$$

Where

$$\langle F | H \rangle = \int_I F(x) H(x) dx$$

Finding the coefficients

- Matrix

$$\mathbf{B} = \begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix}$$

does not depend on G(x)

- – Computed just once for a given basis

Finding the coefficients

- Given a basis $\{B_i(x)\}$
  1. Compute matrix **B**
  2. Compute its inverse $\mathbf{B}^{-1}$
- Given a function G(x) to approximate
  1. Compute dot products

$$\left[\langle G|B_1\rangle \quad \langle G|B_2\rangle \quad \cdots \quad \langle G|B_n\rangle\right]^T$$

  2. … (next slide)

Finding the coefficients

2. Compute coefficients as

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} \langle G | B_1 \rangle \\ \langle G | B_2 \rangle \\ \vdots \\ \langle G | B_n \rangle \end{bmatrix}$$

Orthonormal basis

- Orthonormal basis means

$$\langle B_i | B_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- If basis is orthonormal then

$$\mathbf{B} = \begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = \mathbf{I}$$

Orthonormal basis

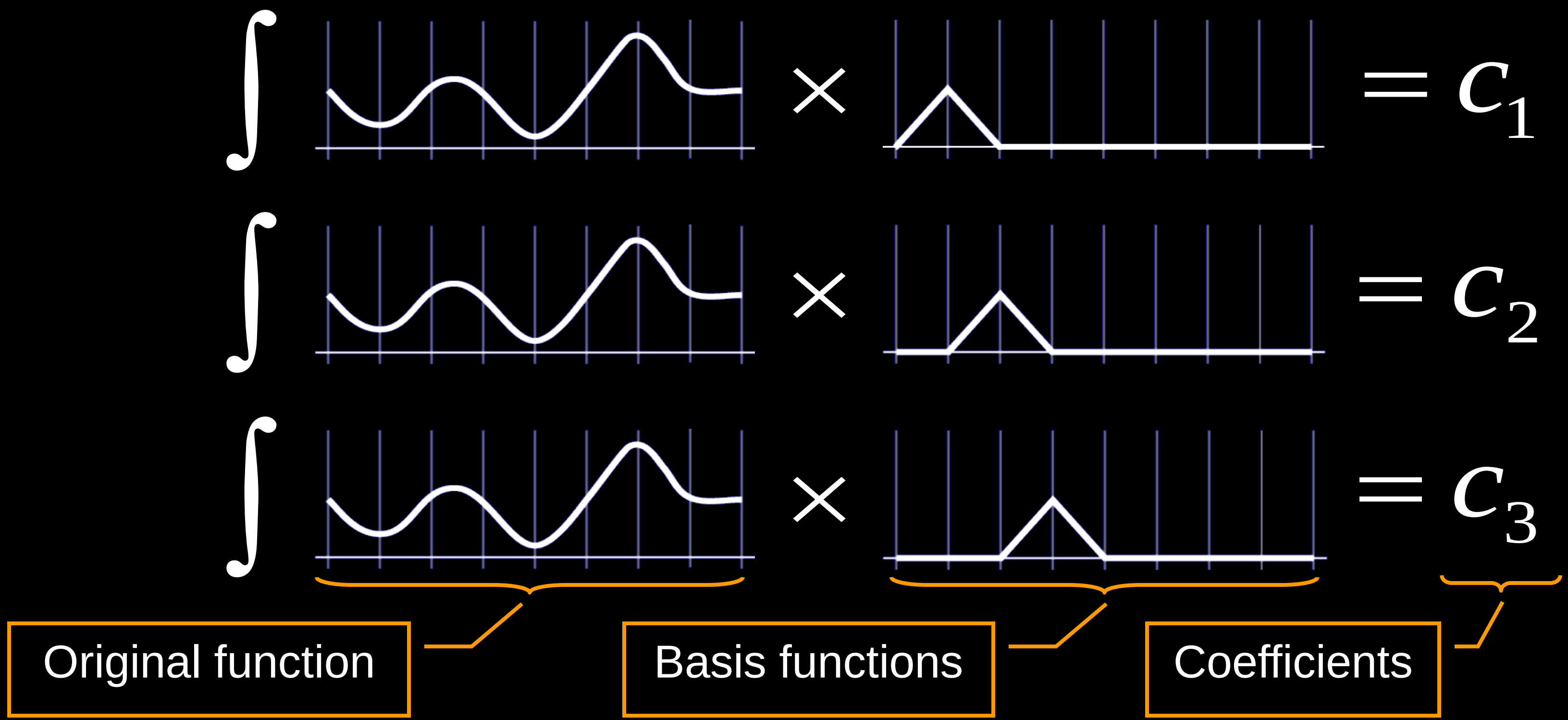
- If the basis is orthonormal, computation of approximation coefficients simplifies to

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle G \mid B_1 \rangle \\ \langle G \mid B_2 \rangle \\ \vdots \\ \langle G \mid B_n \rangle \end{bmatrix}$$

- We want orthonormal basis functions

Orthonormal basis

- Projection: How “similar” is the given basis function to the function we’re approximating



Another reason for orthonormal basis functions

- Intergral of product = dot product of coefficients

$$f(x) = f_i \; B_i(x)$$

$$g(x) = g_i \; B_i(x)$$

$$\int f(x)g(x)dx = f_i \; g_i$$

Application to GI

- Illumination integral

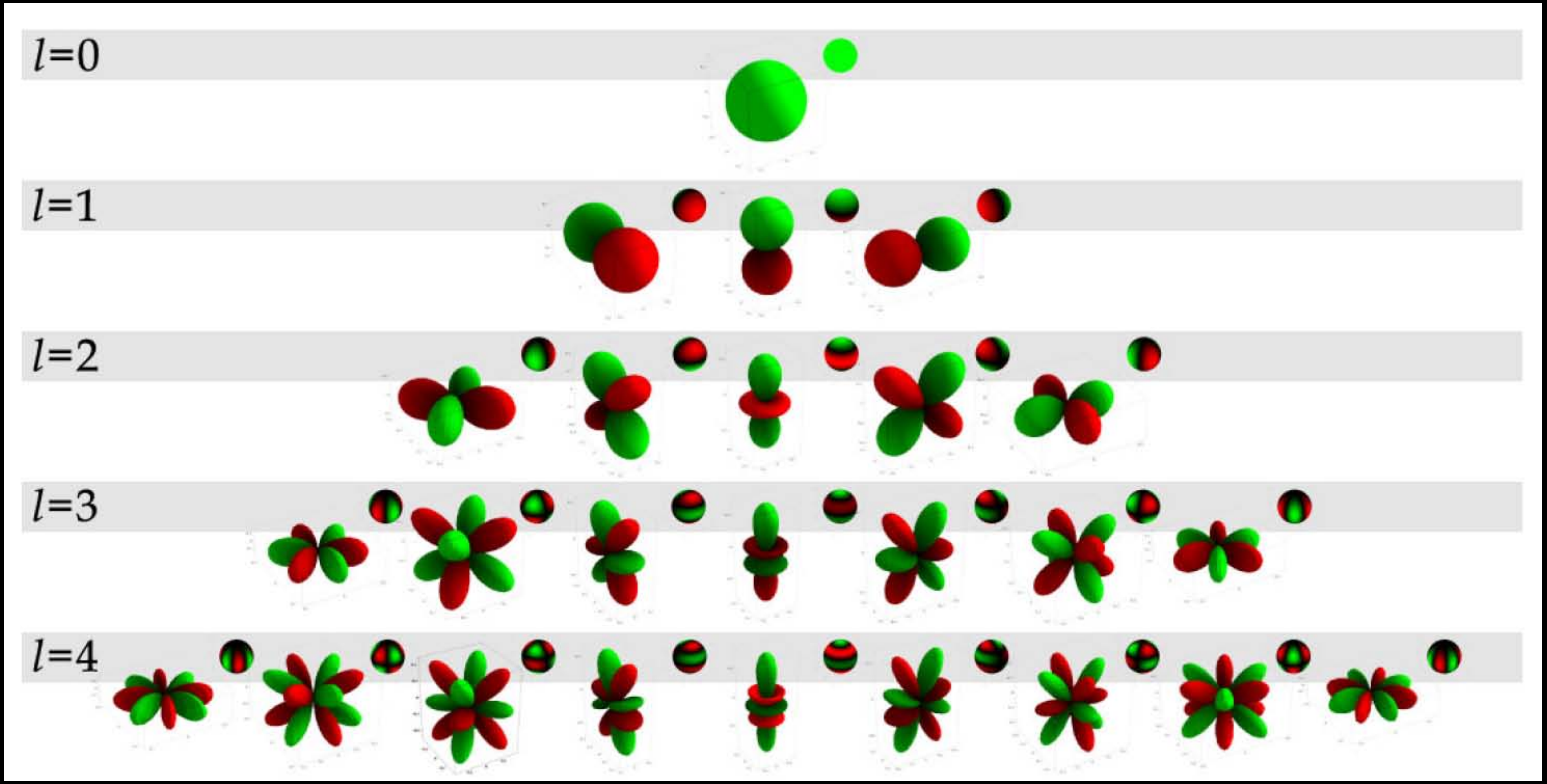
$$L_o = \int L_i(\omega_i)\ \mathrm{BRDF}\ (\omega_i) \cos\theta_i \ d\omega_i$$

Spherical Harmonics

Spherical harmonics

- Spherical function approximation
- Domain *I* = unit sphere *S*
  - directions in 3D
- Approximated function: $G(\theta,\varphi)$
- Basis functions: $Y_i(\theta,\varphi) = Y_{l,m}(\theta,\varphi)$
  - indexing: $i = l\,(l+1) + m$

The SH Functions



Spherical harmonics

$$y_l^m(\theta,\varphi) = \begin{cases} \sqrt{2} K_l^m \cos(m\varphi) P_l^m(\cos\theta), & m > 0 \\ \sqrt{2} K_l^m \sin(-m\varphi) P_l^{-m}(\cos\theta), & m < 0 \\ K_l^0 P_l^0(\cos\theta), & m = 0 \end{cases}$$

- *K* … normalization constant
- *P* … Associated Legendre polynomial
  - Orthonormal polynomial basis on (0,1)
- In general: $Y_{l,m}(\theta,\varphi) = K \,.\, \Psi(\varphi) \,.\, P_{l,m}(\cos\theta)$
  - $Y_{l,m}(\theta,\varphi)$ is separable in θ and φ

Function approximation with SH

$$G(\theta,\varphi)=\sum_{l=0}^{n-1}\sum_{m=-l}^{m=l}c_{l,m}Y_{l,m}(\theta,\varphi)$$

- n…approximation order
- There are n^2 harmonics for order n

Function approximation with SH

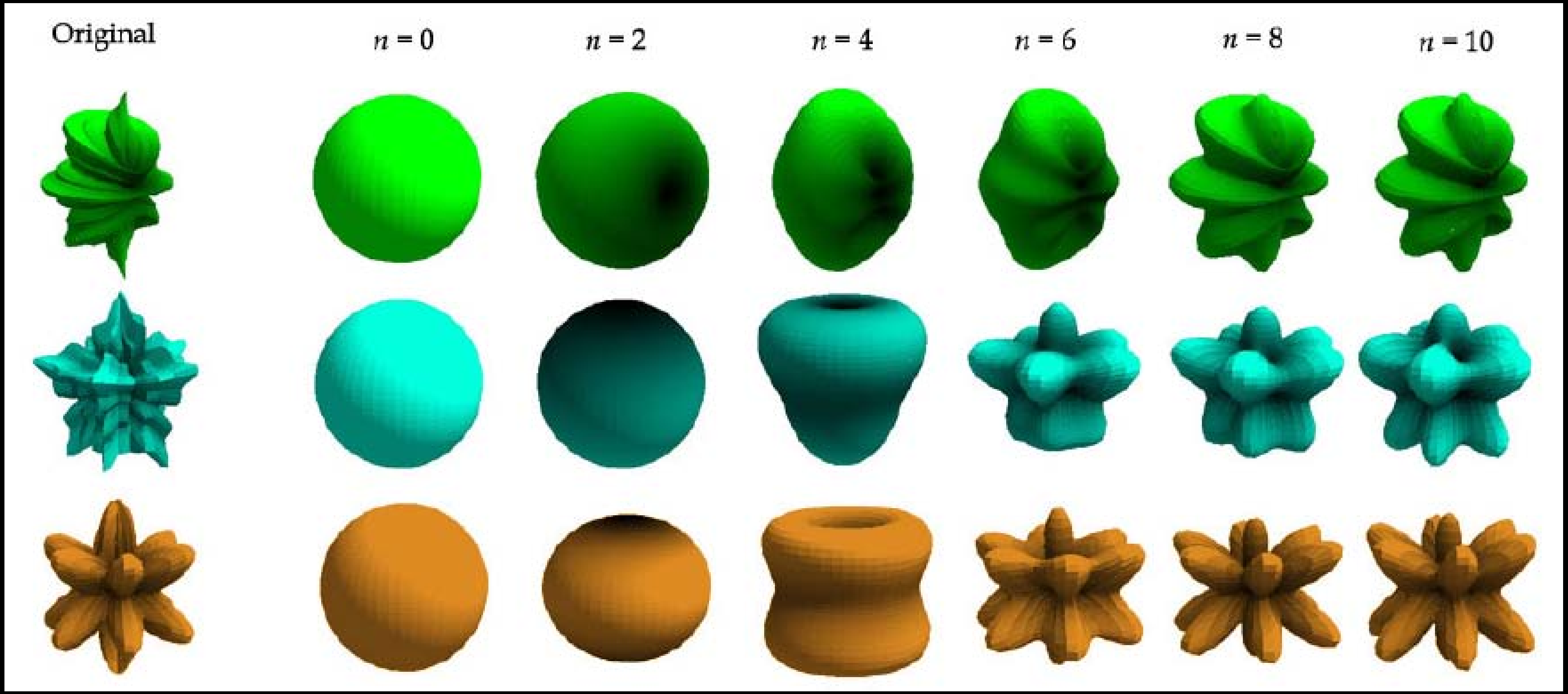
- Spherical harmonics are orthonormal
- Function projection

$$c_{l,m} = \langle G \mid Y_{l,m} \rangle = \int_S G(\omega)\, Y_{l,m}(\omega) d\omega = \int_0^{2\pi} \int_0^{\pi} G(\theta,\varphi)\, Y_{l,m}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

  - Usually evaluated by numerical integration
- Low number of coefficients

  → low-frequency signal

Function approximation with SH



Product integral with SH

- Simplified indexing
  - $Y_i = Y_{l,m}$
  - $i = l\,(l+1) + m$
- Two functions represented by SH

$$F(\omega) = \sum_{i=0}^{n^2} f_i Y_i(\omega)$$

$$G(\omega) = \sum_{i=0}^{n^2} g_i Y_i(\omega)$$

$$\int_S F(\omega) G(\omega) d\omega = \sum_{i=0}^{n^2} f_i g_i$$