Realtime Computer Graphics on GPUs

Math

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Vector Operations

Scalar (Dot) Product

- Definition:

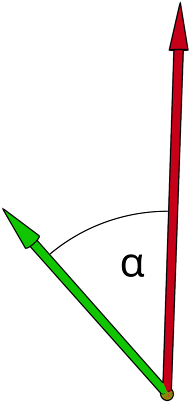
$$\mathbf{p} \cdot \mathbf{q} = \sum_i p_i g_i$$

- Value:

$$\mathbf{p} \cdot \mathbf{q} = ||\mathbf{p}|| \, ||\mathbf{q}|| \cos \alpha$$

- Matrix notation:

$$\mathbf{p} \cdot \mathbf{q} = \mathbf{p}^T \mathbf{q} = [p_0, ... p_{n-1}] \begin{bmatrix} q_0 \\ \vdots \\ q_{n-1} \end{bmatrix}$$



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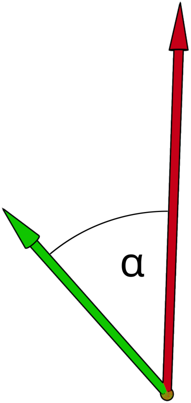

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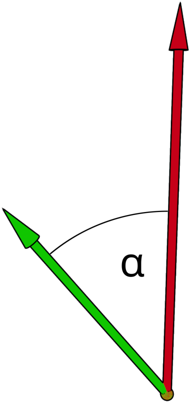


Vector Projection

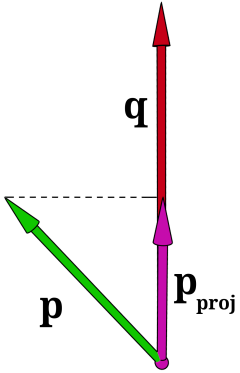
- Projection on another vector:

$$\mathbf{p}_{proj} = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{q}\|}\mathbf{q}$$

- Matrix notation ($\mathbf{q}\mathbf{q}^T$):

$$\mathbf{p}_{proj} = \frac{1}{\|\mathbf{q}\|^2}\begin{bmatrix} q_x^2 & q_xq_y & q_xq_z \\ q_xq_y & q_y^2 & q_yq_z \\ q_xq_z & q_yq_z & q_z^2 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

- Useful for repeated projections, embedding in matrix expressions



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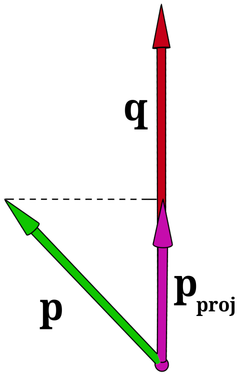

Cross Product

- Definition:

$$\mathbf{p} \times \mathbf{q} = [p_y q_z - p_z q_y, p_z q_x - p_x q_z, p_x q_y - p_y q_x]$$

- As formal determinant:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$$

- Matrix formulation:

$$\mathbf{p} \times \mathbf{q} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

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Cross Product II

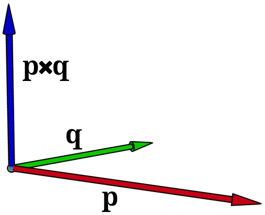
- Perpendicular to $\mathbf{p}, \mathbf{q}$:

$$(\mathbf{p} \times \mathbf{q}) \cdot \mathbf{p} = (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{q} = 0$$

- Size:

$$\|\mathbf{p} \times \mathbf{q}\| = \|\mathbf{p}\| \|\mathbf{q}\| \sin \alpha$$

- Follows *right hand rule*



Rotations

2D ROTATION

- Basic expression:

$$\begin{aligned} x' &= x\cos\alpha - y\sin\alpha \\ y' &= x\sin\alpha + y\cos\alpha \end{aligned}$$

- Matrix notation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Complex exponential:

$$[x, y] \Rightarrow z = x + iy$$

  - Multiply by $e^{i\alpha} = \cos\alpha + i\sin\alpha$
  - Inverse rotation by complex conjugate

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Elementary Rotations in 3D

$$\mathbf{R_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\mathbf{R_y} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\mathbf{R_z} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Around Arbitrary Axis

- Axis $\mathbf{a}$, angle θ, point $\mathbf{p}$, rotated point p':

$$\|\mathbf{a}\| = 1$$

- Project $\mathbf{p}$ onto axis:

$$\begin{aligned} \mathbf{p}_{proj} &= (\mathbf{a} \cdot \mathbf{p})\mathbf{a} \\ R_{\mathbf{a},\theta}\mathbf{p}_{proj} &= \mathbf{p}_{proj} \end{aligned}$$

- Perpendicular component:

$$\begin{aligned} \mathbf{p}_{perp} &= \mathbf{p} - (\mathbf{a} \cdot \mathbf{p})\mathbf{a} \\ \|\mathbf{p}_{perp}\| &= \|\mathbf{p}\| \sin \alpha \end{aligned}$$

- Cross product with axis:

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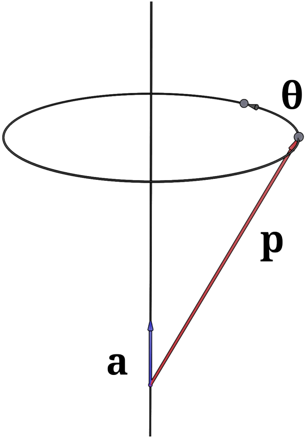

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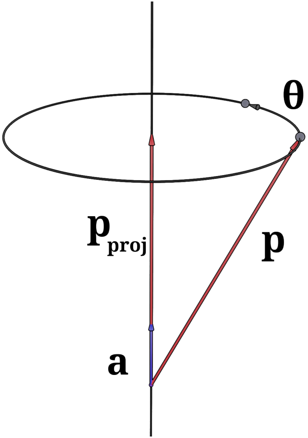

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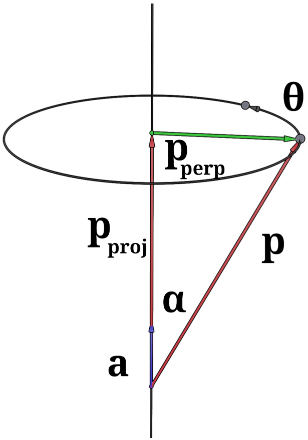

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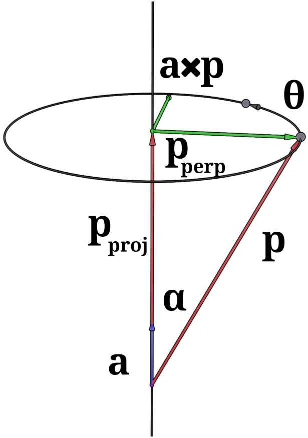

ROTATION AROUND ARBITRARY AXIS

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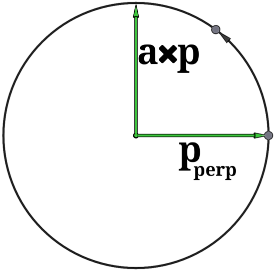

Rotation Around Arbitrary Axis II

- Final rotated position:

$$\begin{aligned} \mathbf{p}'_{perp} &= \mathbf{p}_{perp}\cos\theta + (\mathbf{a}\times\mathbf{p})\sin\theta \\ \mathbf{p}' &= \mathbf{p}'_{perp} + \mathbf{p}_{proj} \end{aligned} \tag{1}$$

- Matrix representation:

$$\begin{aligned} \mathbf{p}'_{perp} &= [\mathbf{p} - (\mathbf{a}\cdot\mathbf{p})\mathbf{a}]\cos\theta + (\mathbf{a}\times\mathbf{p})\sin\theta \\ &= \mathbf{p}\cos\theta + (\mathbf{a}\times\mathbf{p})\sin\theta + \mathbf{a}(\mathbf{a}\cdot\mathbf{p})(1-\cos\theta) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{p}\cos\theta + \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}\mathbf{p}\sin\theta \\ &+ \begin{bmatrix} A_x^2 & A_xA_y & A_xA_z \\ A_xA_y & A_y^2 & -A_yA_z \\ A_xA_z & A_yA_z & A_z^2 \end{bmatrix}\mathbf{p}(1-\cos\theta) \end{aligned}$$

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Rotation Around Arbitrary Axis III

- Final matrix form:

$$\begin{bmatrix} c+(1-c)A_x^2 & (1-c)A_xA_y - sA_z & (1-c)A_xA_z + sA_y \\ (1-c)A_xA_y + sA_z & c+(1-c)A_y^2 & (1-c)A_yA_z - sA_x \\ (1-c)A_xA_z - sA_y & (1-c)A_yA_z + sA_x & c+(1-c)A_z^2 \end{bmatrix}$$

Euler Angles

- arbitrary rotation decomposed into three components
- Leonard Euler (1707-1783)
- 3 angles – 3 elementary rotations
- order of rotations important (x-y-z, roll-pitch-yaw, z-x-z, ...)
  - intrinsic vs. extrinsics

EULER ANGLES II

Disadvantages:

- Problematic interpolation between two orientations
- Gimbal lock – not as severe in SW as in HW (Apollo)

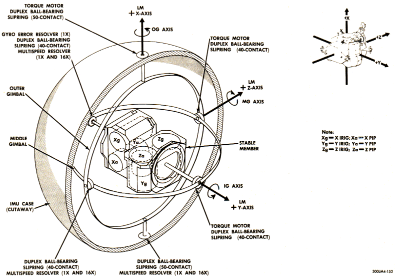

Figure 2.1-24. IMU Gimbal Assembly

Quaternions

- Sir William Rowan Hamilton, 16 Oct 1843 (Dublin)
- generalization of complex numbers in 4D space
- usage in graphics since 1985 (Shoemake)
- $\mathbf{q} = (\mathbf{v}, w) = ix + jy + kz + w = \mathbf{v} + w$
- imaginary part $v = (x, y, z) = ix + jy + kz$
- $i^2 = j^2 = k^2 = -1, jk = -kj = i, ki = -ik = j, ij = -ji = k$

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Quaternions - Why 4D?

- 3D – what is $(ij) =?$

$$(i)(x+iy+jz) = -y+ix+(ij)z$$

- We need to introduce $ij = k$

$$(i)(ix+jy+kz+w) = -x+iw-jz+ky$$

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Quaternion Operations

- addition $(\mathbf{v}_1, w_1) + (\mathbf{v}_2, w_2) = (\mathbf{v}_1 + \mathbf{v}_2, w_1 + w_2)$
- multiplication $\mathbf{qr} = (\mathbf{v}_q \times \mathbf{v}_r + w_r\mathbf{v}_q + w_q\mathbf{v}_r, w_q w_r - \mathbf{v}_q \cdot \mathbf{v}_r)$
- multiplication by a scalar $s\mathbf{q} = (0, s)(\mathbf{v}, w) = (s\mathbf{v}, sw)$
- conjugation $(\mathbf{v}, w)^* = (-\mathbf{v}, w)$
- unit $id = (\mathbf{0}, 1)$
- norm (squared absolute value) $||\mathbf{q}||^2 = n(\mathbf{q}) = \mathbf{qq}^* = x^2 + y^2 + z^2 + w^2$
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Quaternion Operations II

- unit quaternion can be expressed by goniometry as $\mathbf{q} = (\mathbf{u}_q \sin\theta, \cos\theta)$
- for some unit 3D vector $\mathbf{u}_q$ it represents a rotation (orientation) in 3D
  - ambiguity: both $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation!
- identity (zero rotation): $(0, 1)$
- power, exponential, logarithm:
  $\mathbf{q} = \mathbf{u}_q \sin\theta + \cos\theta = exp(\theta\mathbf{u}_q), log\mathbf{q} = \theta\mathbf{u}_q$
  $\mathbf{q}^t = (\mathbf{u}_q \sin\theta + cos\theta)^t = exp(t\theta\mathbf{u}_q) = \mathbf{u}_q \sin t\theta + \cos t\theta$

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- for some unit 3D vector $\mathbf{u}_q$ it represents a rotation (orientation) in 3D
  - ambiguity: both $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation!
- identity (zero rotation): $(0, 1)$
- power, exponential, logarithm:
  $\mathbf{q} = \mathbf{u}_q \sin\theta + \cos\theta = exp(\theta\mathbf{u}_q), log\mathbf{q} = \theta\mathbf{u}_q$
  $\mathbf{q}^t = (\mathbf{u}_q \sin\theta + cos\theta)^t = exp(t\theta\mathbf{u}_q) = \mathbf{u}_q \sin t\theta + \cos t\theta$

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Quaternion Rotations

- unit quaternion $\mathbf{q} = (\mathbf{u}_q \sin\theta, \cos\theta)$
  - $\mathbf{u}_q$ axis of rotation
  - θ angle
- vector (point) in 3D: $\mathbf{p} = [p_x, p_y, p_z, 0]$
- rotation of vector (point) $\mathbf{p}$ around $\mathbf{u}_q$ by angle 2θ
  $\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \mathbf{q}\mathbf{p}\mathbf{q}^*$

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Quaternion Rotations - Why 2θ

- Rotate by i from left:

$$(i)(w + ix + jy + kz) = -x + iw - jz + ky$$

- Rotate by i from right:

$$(w + ix + jy + kz)(i) = -x + iw + jz - ky$$

- Rotate by i from both sides:

$$(i)(w + x + jy + kz)(i) = -w - ix + jy + kz$$

- Prevent rotation in w:

$$(i)(w+ix+jy+kz)(i^{-1}) = (i)(ix+jy+kz+w)(-i) = w-ix+jy+kz$$

- To prevent the second 4D rotation we rotated twice around the first axis.

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SPHERICAL LINEAR INTERPOLATION – SLERP

- two quaternions $\mathbf{q}$ and $\mathbf{r}$ ($\mathbf{q} \cdot \mathbf{r} \geq 0$, else take $-\mathbf{q}$)
- real parameter $0 \leq t \leq 1$
- interpolated quaternion $slerp(\mathbf{q}, \mathbf{r}, t) = \mathbf{q}(\mathbf{q}^*\mathbf{r})^t$

$$slerp(q, r, t) = \frac{\sin(\theta(1-t))}{\sin\theta}\mathbf{q} + \frac{\sin\theta t}{\sin\theta}\mathbf{r}$$

- the shortest spherical arc between $\mathbf{q}$ and $\mathbf{r}$

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Quaternion from Two Vectors

- two vectors s and t:
  1. normalization of s, t
  2. unit rotation axis $u = (s \times t)/\|s \times t\|$
  3. angle between s and t: $s \cdot t = \cos\theta$
- Identities to prevent trigonometry:

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \quad (2)$$

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$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad (4)$$

- Final quaternion:

$$q = \left(norm(u)\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) = \left(\frac{s \times t}{\sqrt{2(1+s\cdot t)}}, \sqrt{\frac{1+s\cdot t}{2}}\right)$$

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SUMMARY

rotational matrix

+ HW support, efficient point/vector transformation
– memory (float[9]), other operations are not so efficient

rotational axis and angle

+ memory (float[4] or float[6]), similar to quaternion
– inefficient composition and interpolation

quaternion

+ memory (float[4]), composition, interpolation
– inefficient point/vector transformation

Affine and Projective Spaces

AFFINNE AND PROJECTIVE SPACES

Affine space:

- Set V of vectors and set P of points
- Affine transformations can be represented by matrix

Projective space:

- Homogeneous coordinates
- All lines intersect (space contains infinity)
- Affine and projective transformations can be represented by matrix

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Homogeneous Coordinates

- homogeneous coordinate vector $[x, y, z, w]$
- transformation: multiplying by a 4×4 matrix
- homogeneous matrix is able to translate and to do perspective projections
- from homogeneous coordinates $[x, y, z, w]$ into Cartesian coordinates: by division $(w \neq 0)[\ x/w,\ \ y/w,\ \ z/w]$
- coordinate vector $[x, y, z, 0]$ – point in infinity
- from Cartesian coordinates to homogeneous: trivial extension $[x, y, z] \dots [x, y, z, 1]$

Transformation Matrix

$$\mathbf{Ap} = \left[\begin{array}{ccc|c} & \mathbf{M} & & \mathbf{T} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

- $\mathbf{T}$ defines translation
- $\mathbf{M}$ defines:
  - rotation
  - scaling

$$\mathbf{M}_{scale} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

  - shear

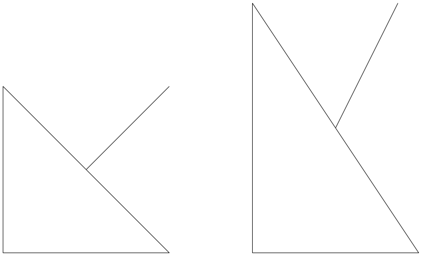
$$\mathbf{M}_{shear} = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - and their combinations

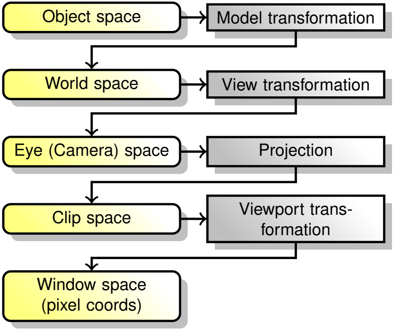
Normal Vector Transformation

- Only orientation change is valid transformation for normals
- Tangents (t) remain valid:

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{t}=0 &\Rightarrow \mathbf{n}'\cdot\mathbf{t}'=(\mathbf{G}\mathbf{n})\cdot(\mathbf{M}\mathbf{t})=0\\
(\mathbf{G}\mathbf{n})\cdot(\mathbf{M}\mathbf{t}) &= (\mathbf{G}\mathbf{n})^T(\mathbf{M}\mathbf{t})\\
&= \mathbf{n}^T\mathbf{G}^T\mathbf{M}\mathbf{t}\\
&\Rightarrow \mathbf{G}=(\mathbf{M}^{-1})^T
\end{aligned}$$



Transformations for Rendering Pipeline



LOOKAT CAMERA MATRIX

- Camera position (eye) $\mathbf{e}$
- Lookat point $\mathbf{p}$
- Up vector $\mathbf{u}$

$$
\begin{aligned}
\mathbf{v} &= norm(\mathbf{e} - \mathbf{p}) \\
\mathbf{n} &= norm(\mathbf{v} \times \mathbf{u})
\end{aligned}
\tag{5}
$$

Matrix which transforms camera into its position:

$$
\mathbf{TR} = \begin{bmatrix} 1 & 0 & 0 & e_x \\ 0 & 1 & 0 & e_y \\ 0 & 0 & 1 & e_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & u_x & v_x & 0 \\ n_y & u_y & v_y & 0 \\ n_z & u_z & v_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
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LookAt Camera Matrix II

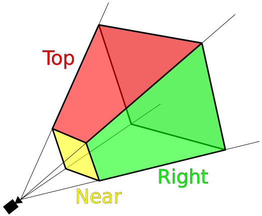
World view needs to be transformed by its inverse:

$$
\begin{aligned}
(\mathbf{TR})^{-1} &= \mathbf{R^{-1}T^{-1}} = \mathbf{R^{T}T^{-1}} = \begin{bmatrix} n_x & n_y & n_z & 0 \\ u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} n_x & n_y & n_z & -(n \cdot e_x) \\ u_x & u_y & u_z & -(u \cdot e_y) \\ v_x & v_y & v_z & -(v \cdot e_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Perspective Projection

Point p projection: $x = -\frac{n}{p_z} p_x$ and $y = -\frac{n}{p_z} p_y$

$$\mathbf{P}_{frustum} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2nf}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Perspective correct interpolation