Aliasing And Anti-Aliasing
Sampling and Reconstruction

An Introduction

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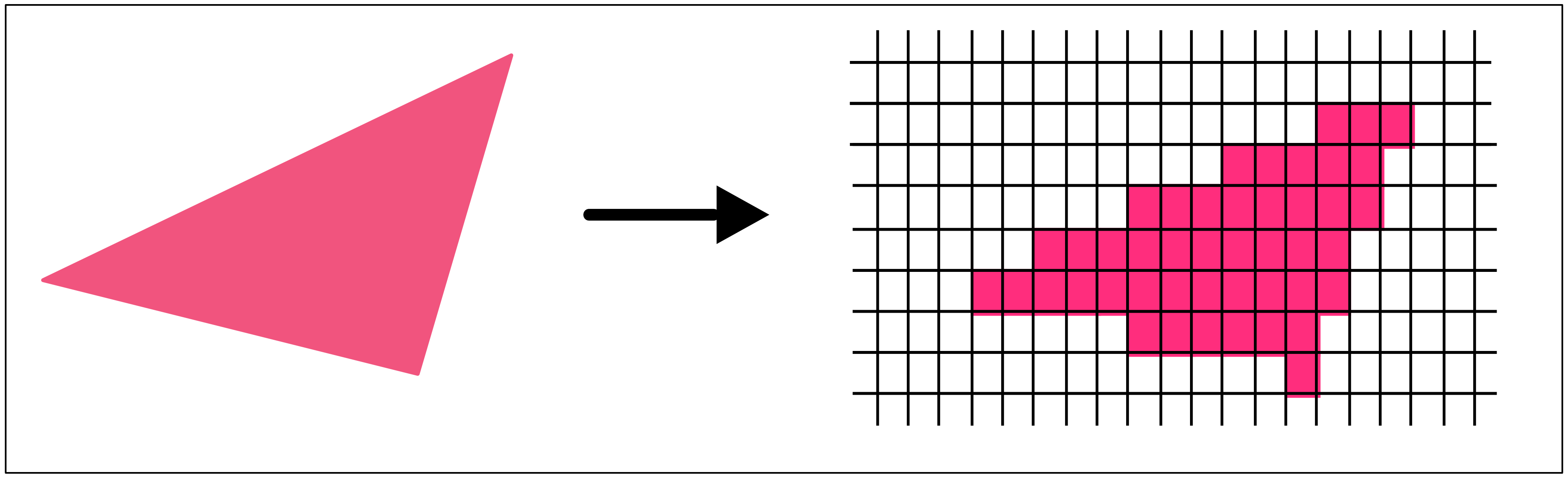
Overview

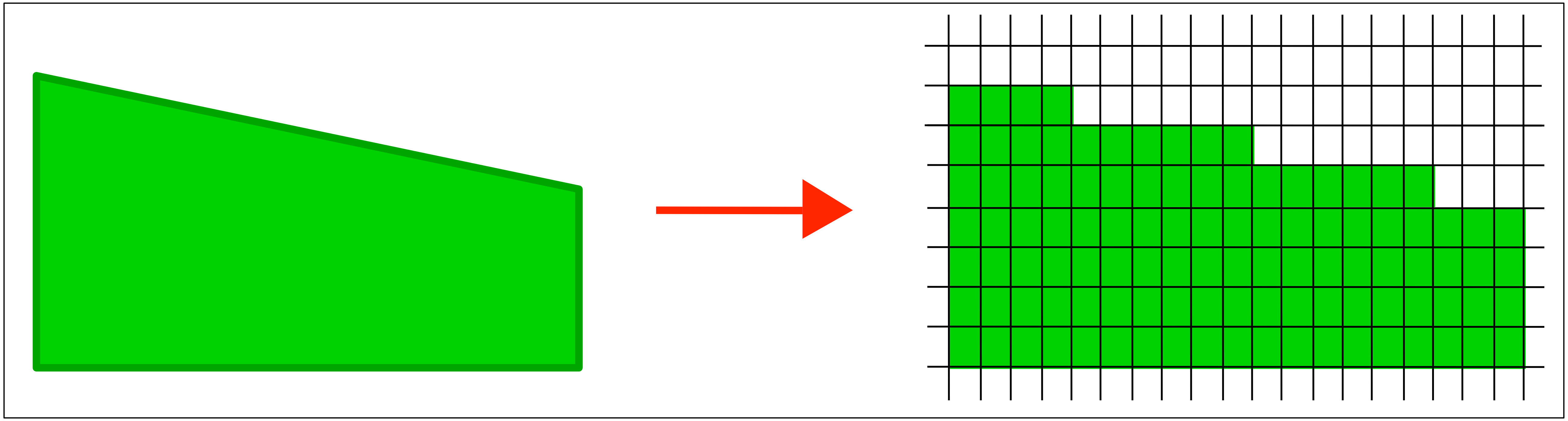
- Intro - Aliasing
  - Problem definition, Examples
- Ad-hoc Solutions
- Sampling theory
  - Fourier transform
  - Convolution
- Reconstruction
  - Sampling theorem
  - Reconstruction in theory and practice

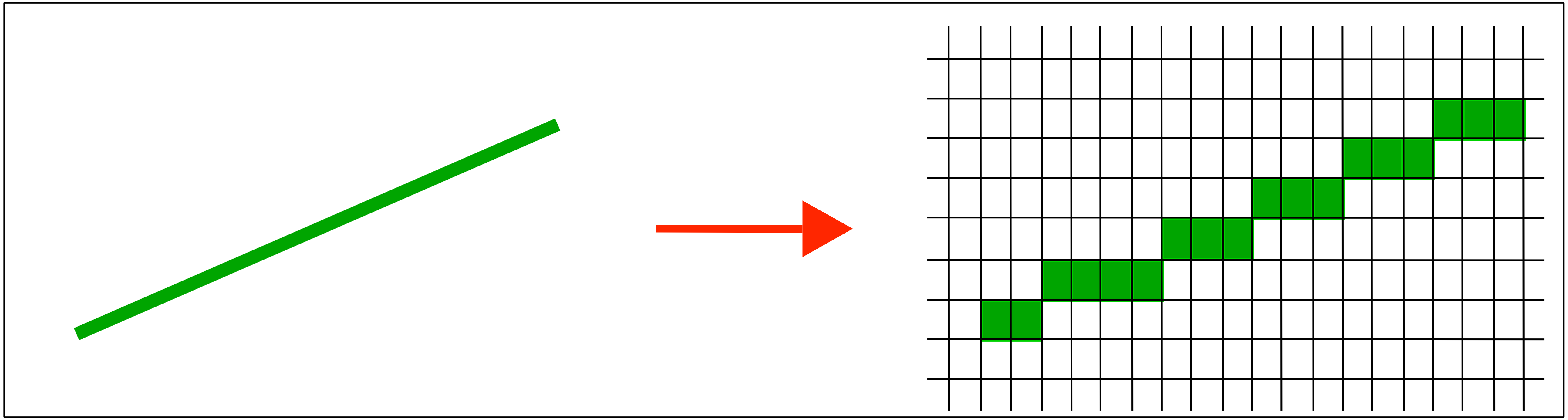
Aliasing - a Common Problem



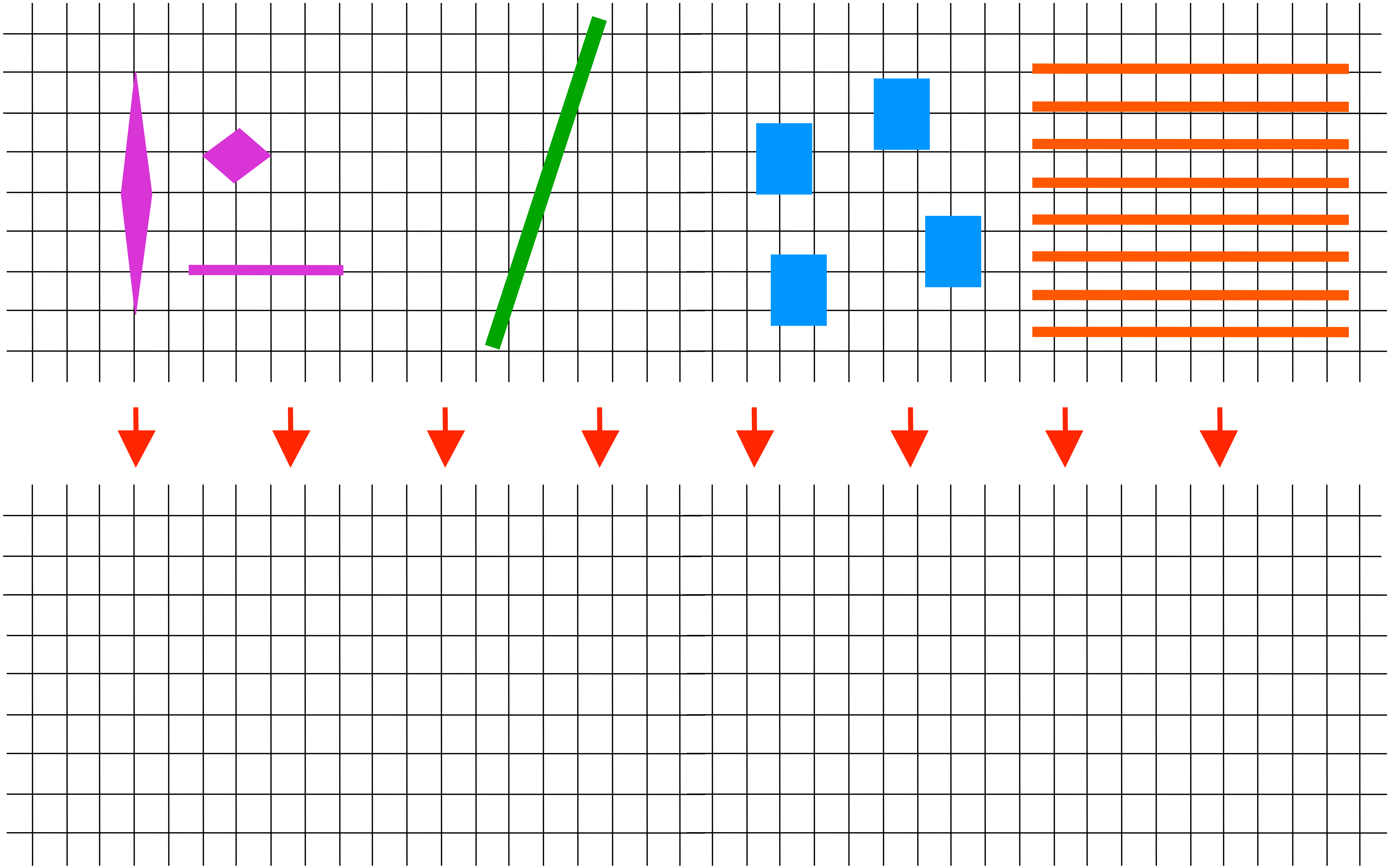
- Errors incurred during analog-digital conversions:
  - Geometric (pixel artefacts, „jaggies“)
  - Colour quality
  - Errors in animation sequences

Aliasing: „Jaggies“

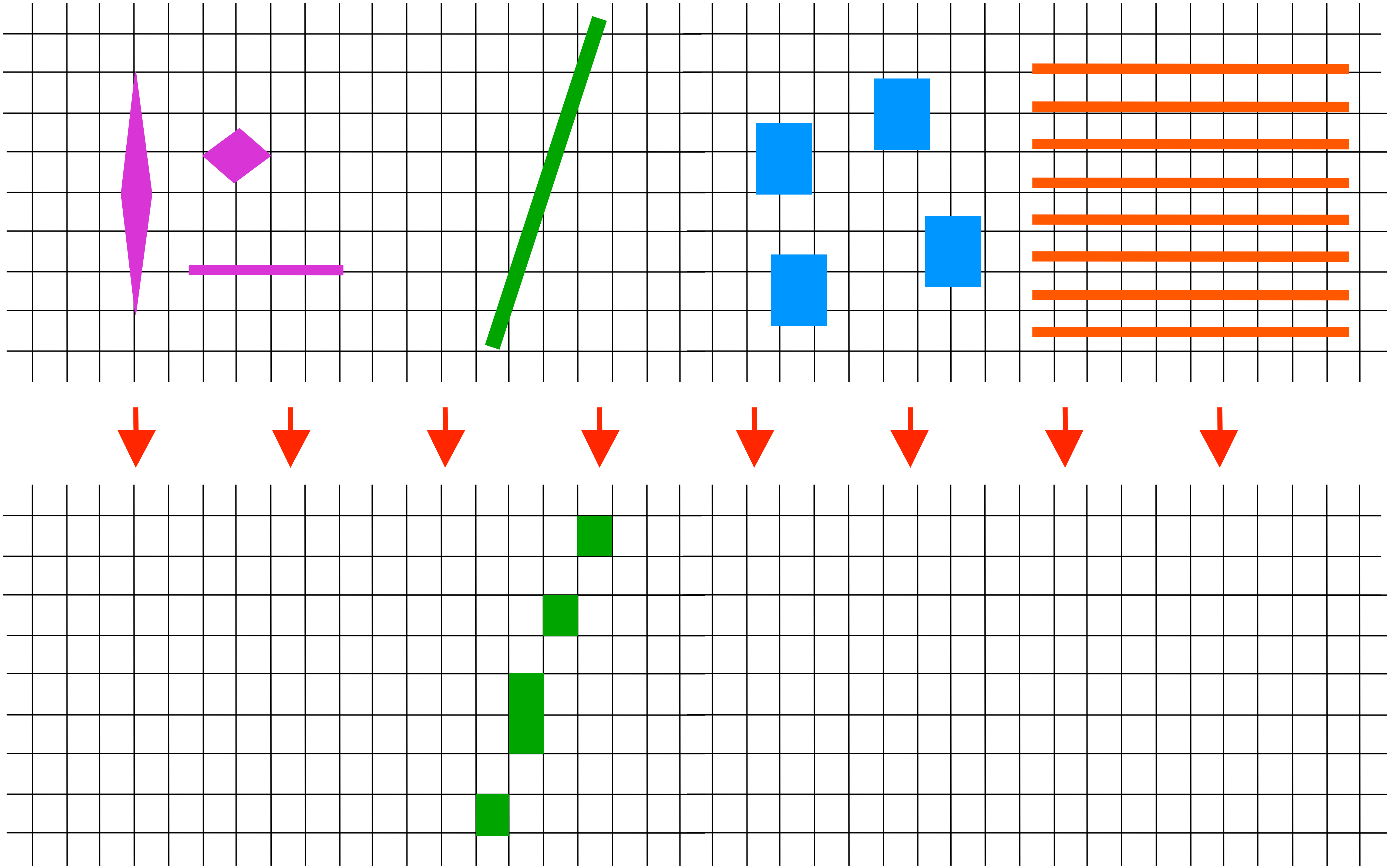




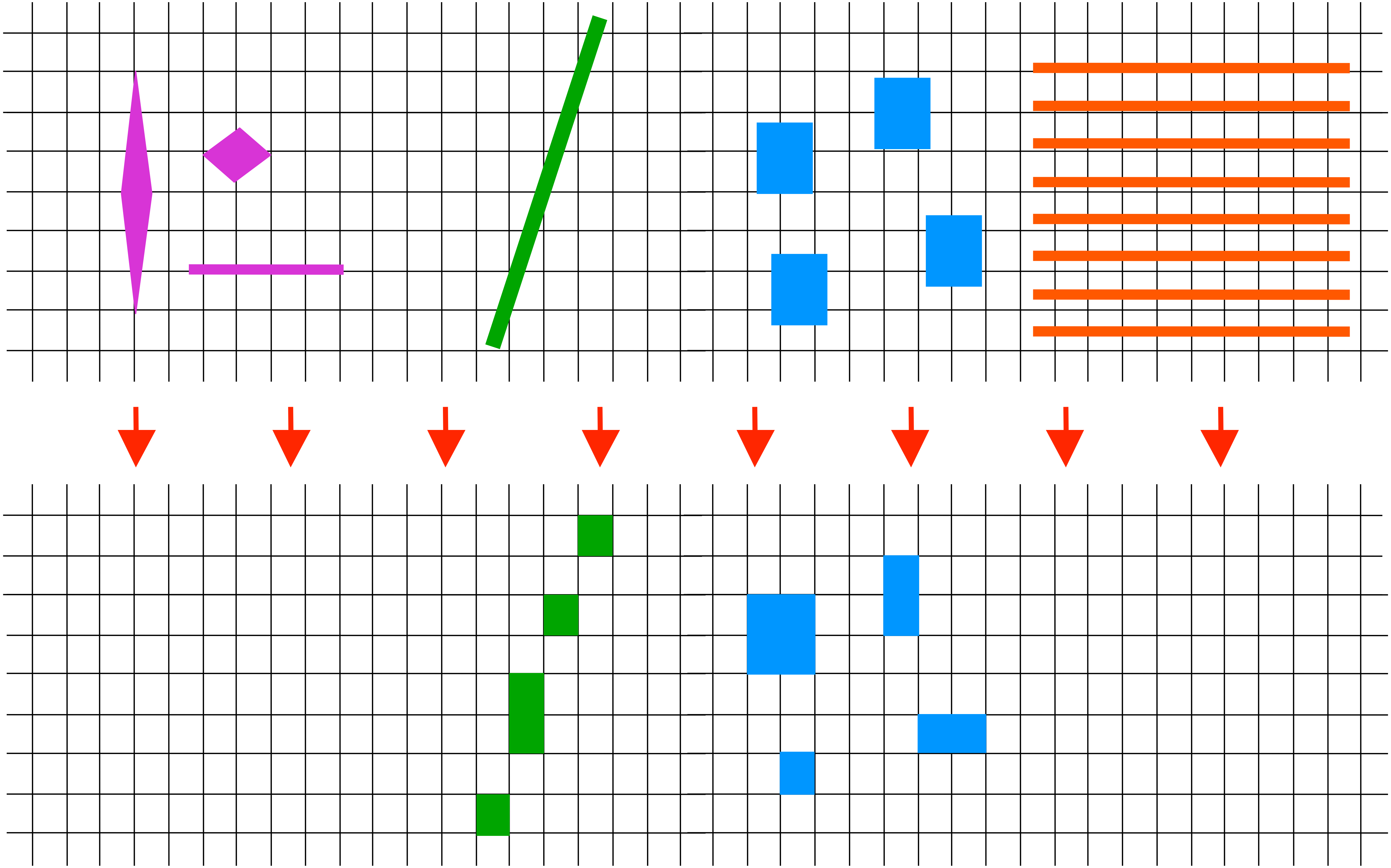
Aliasing: Sampling Errors



Aliasing: Sampling Errors



Aliasing: Sampling Errors



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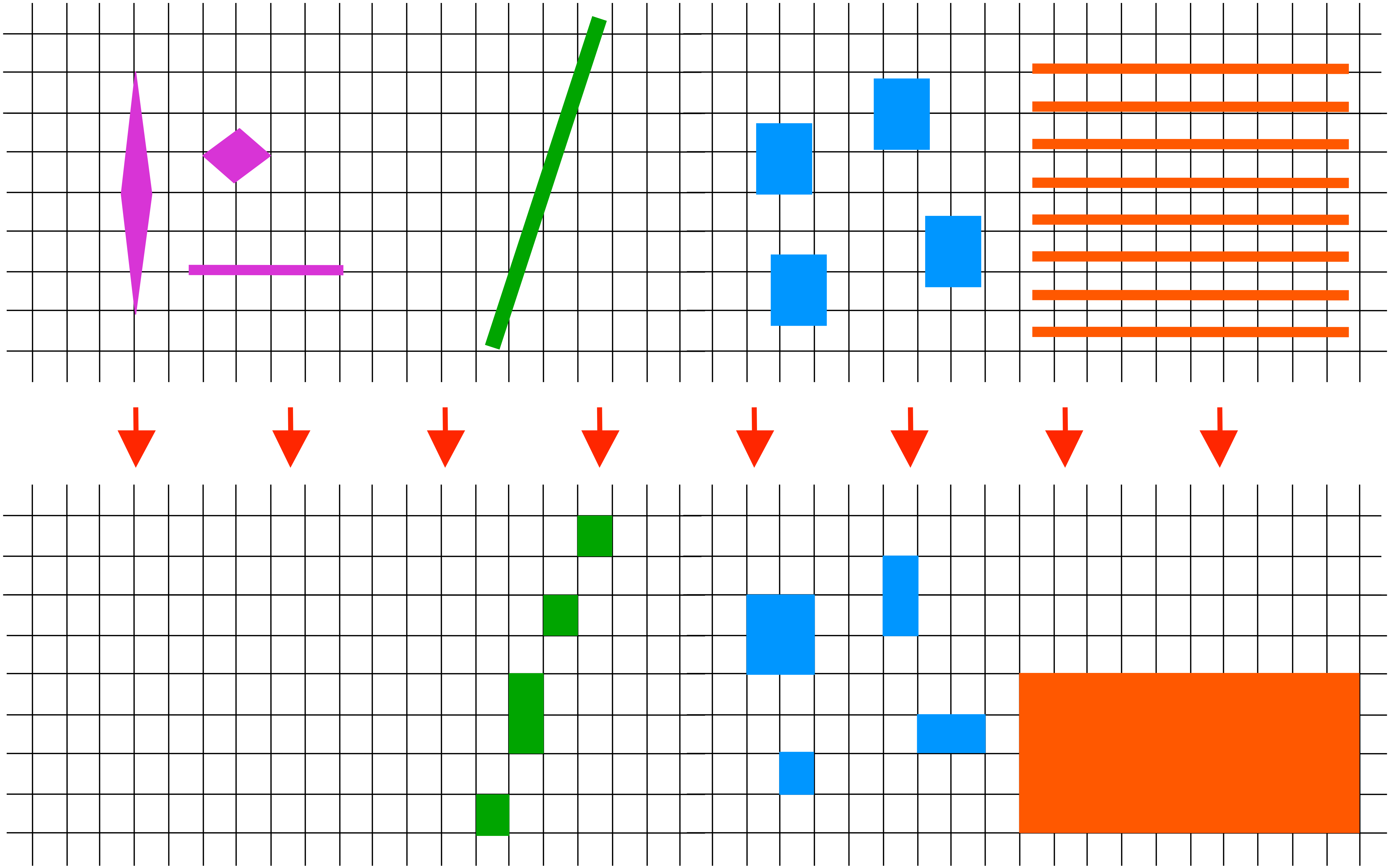
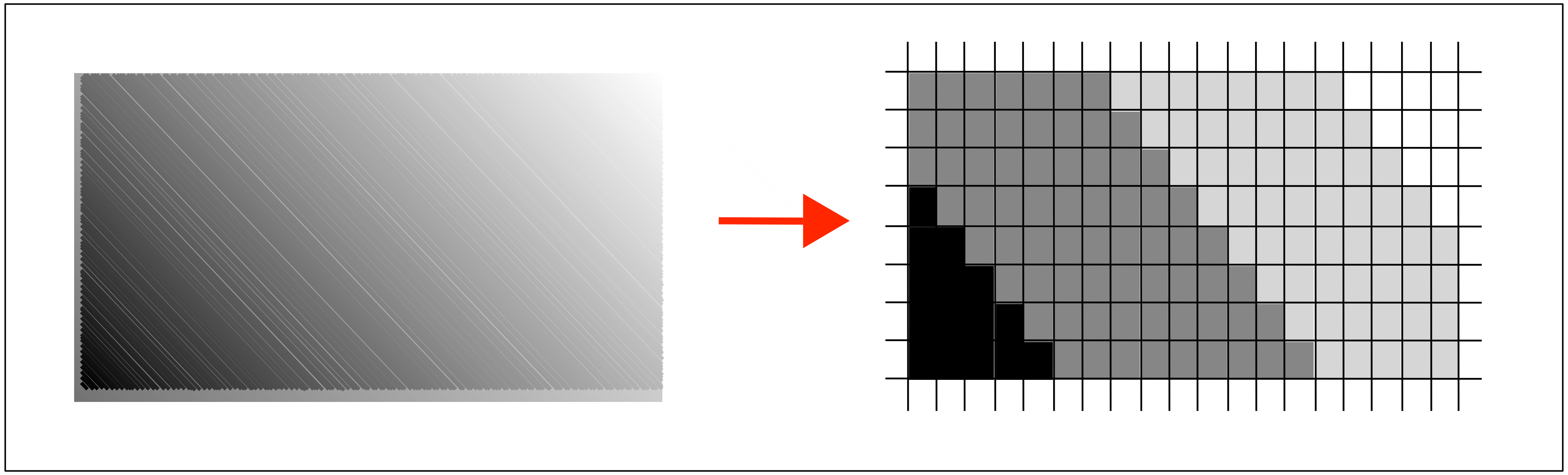
Aliasing: Sampling Errors

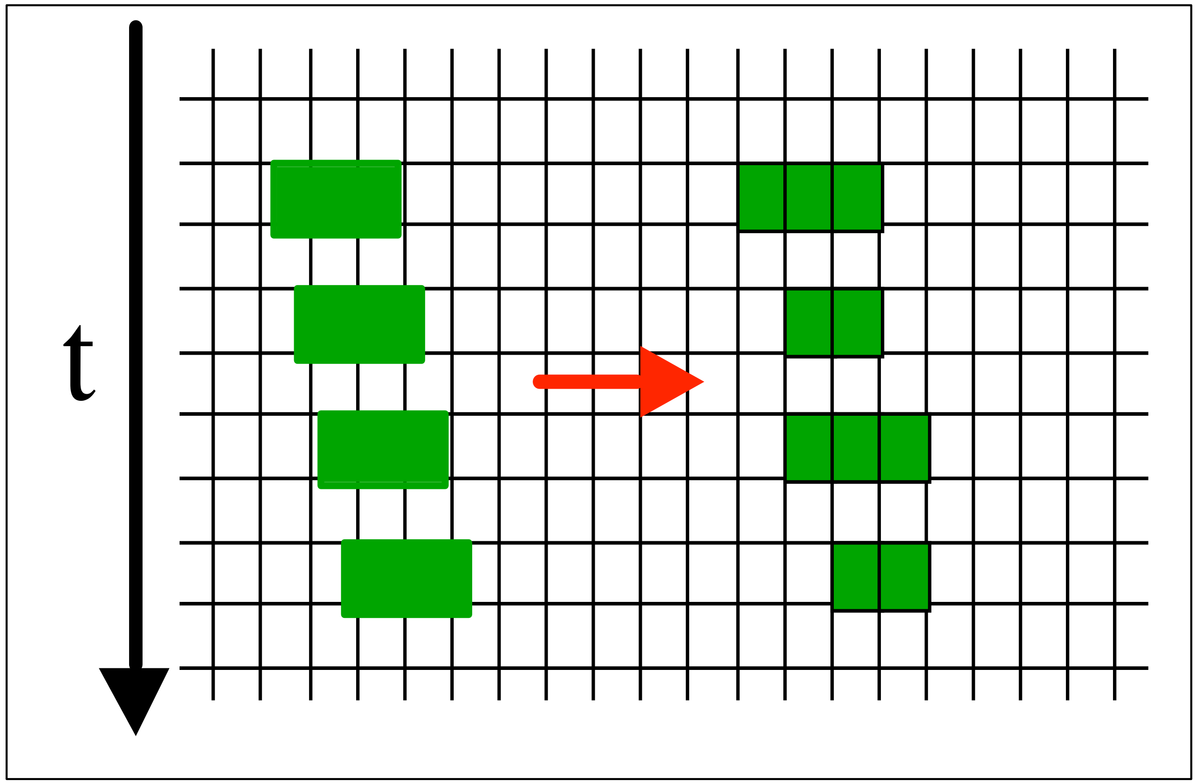


Aliasing: Colour Ramps

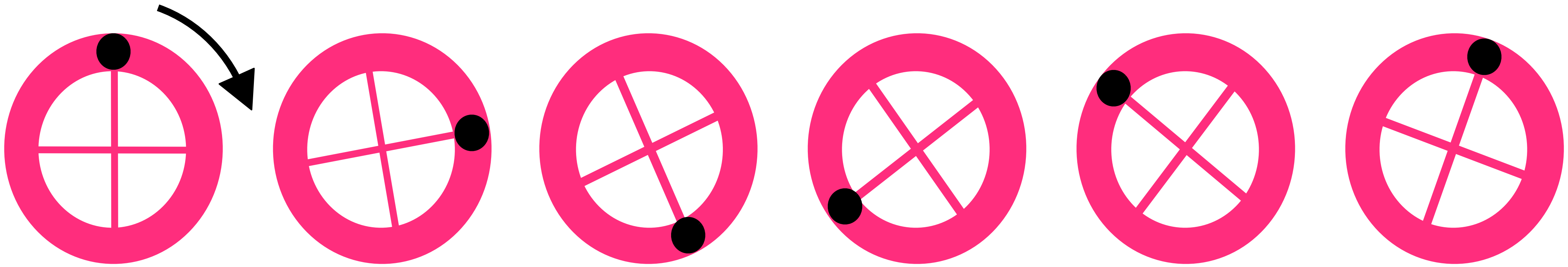


Animation Aliasing

- Jumpy images
- "worming“

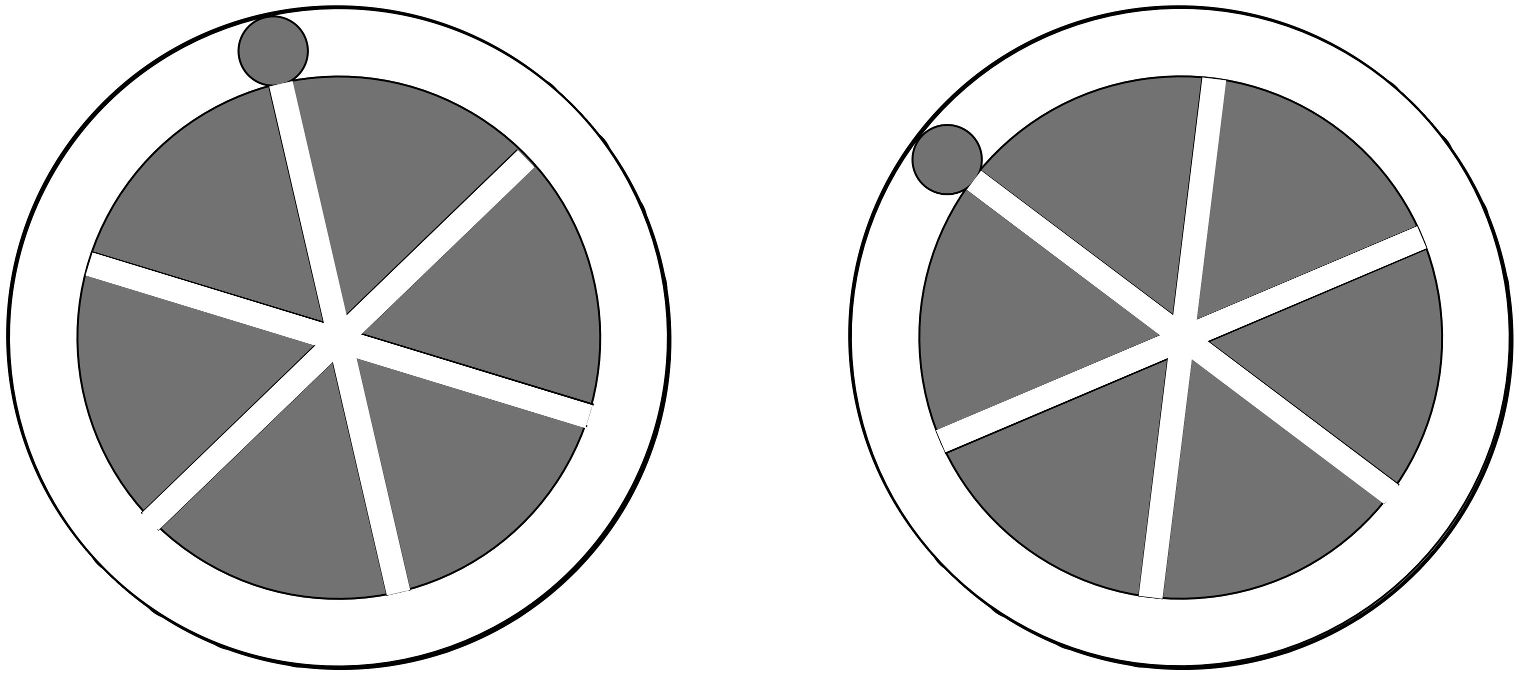


- Wheels turning backwards



Wheels turning backwards

Wheels turning backwards



Solution Strategies

- “More effort”
  - Ad-hoc solutions
  - Higher resolution
  - Higher colour depth
  - Faster image refresh
- “Use your brain”
  - Understanding the problem
  - Efficient counter-measures

Solution Strategies

Oft aufwendig,
nicht immer
zielführend,
manchmal
unmöglich

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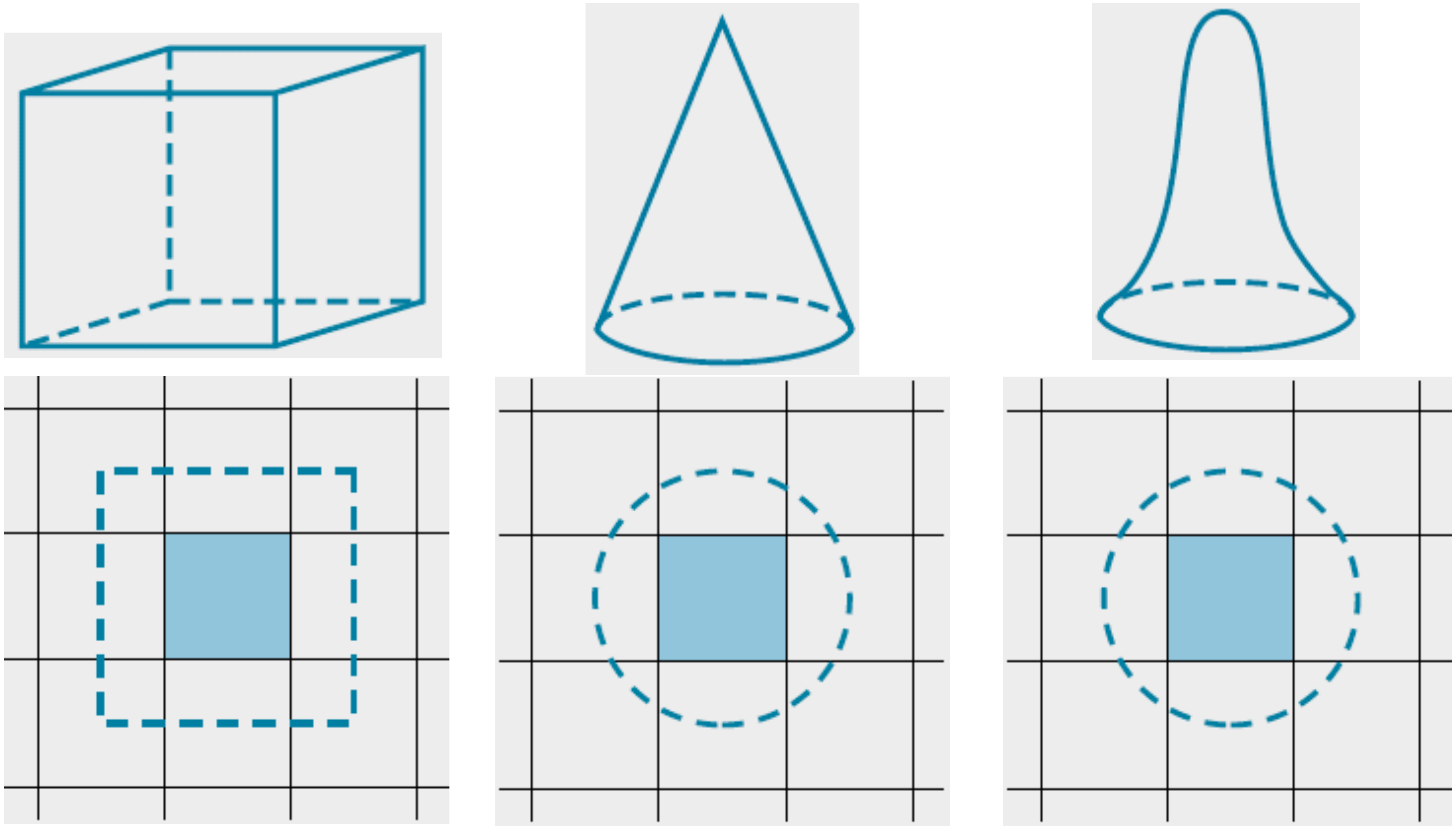
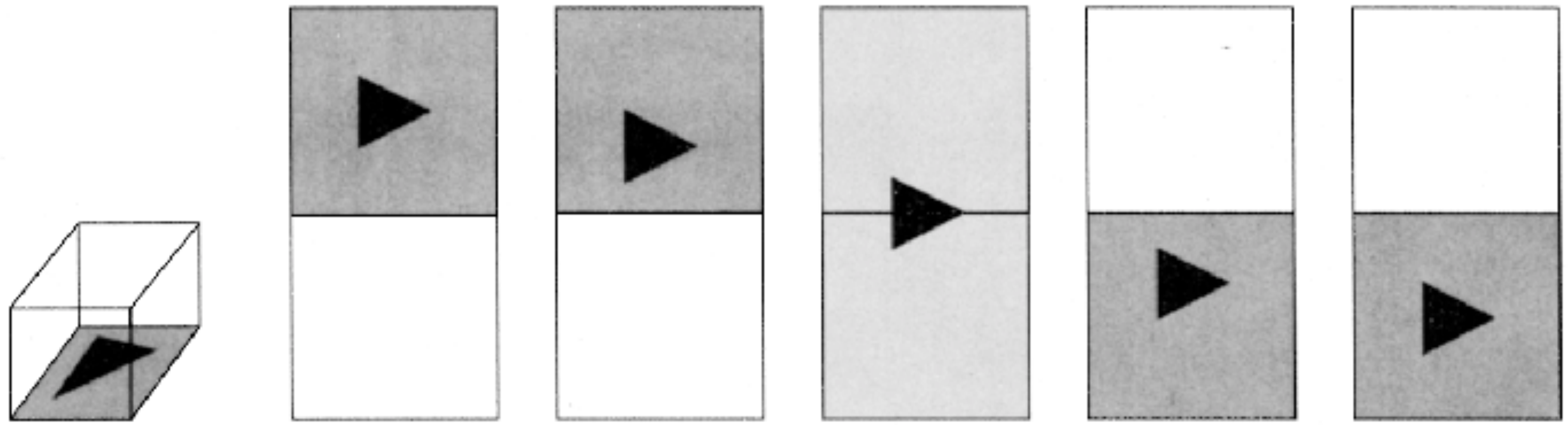
machbar

Simple Splatting Kernels

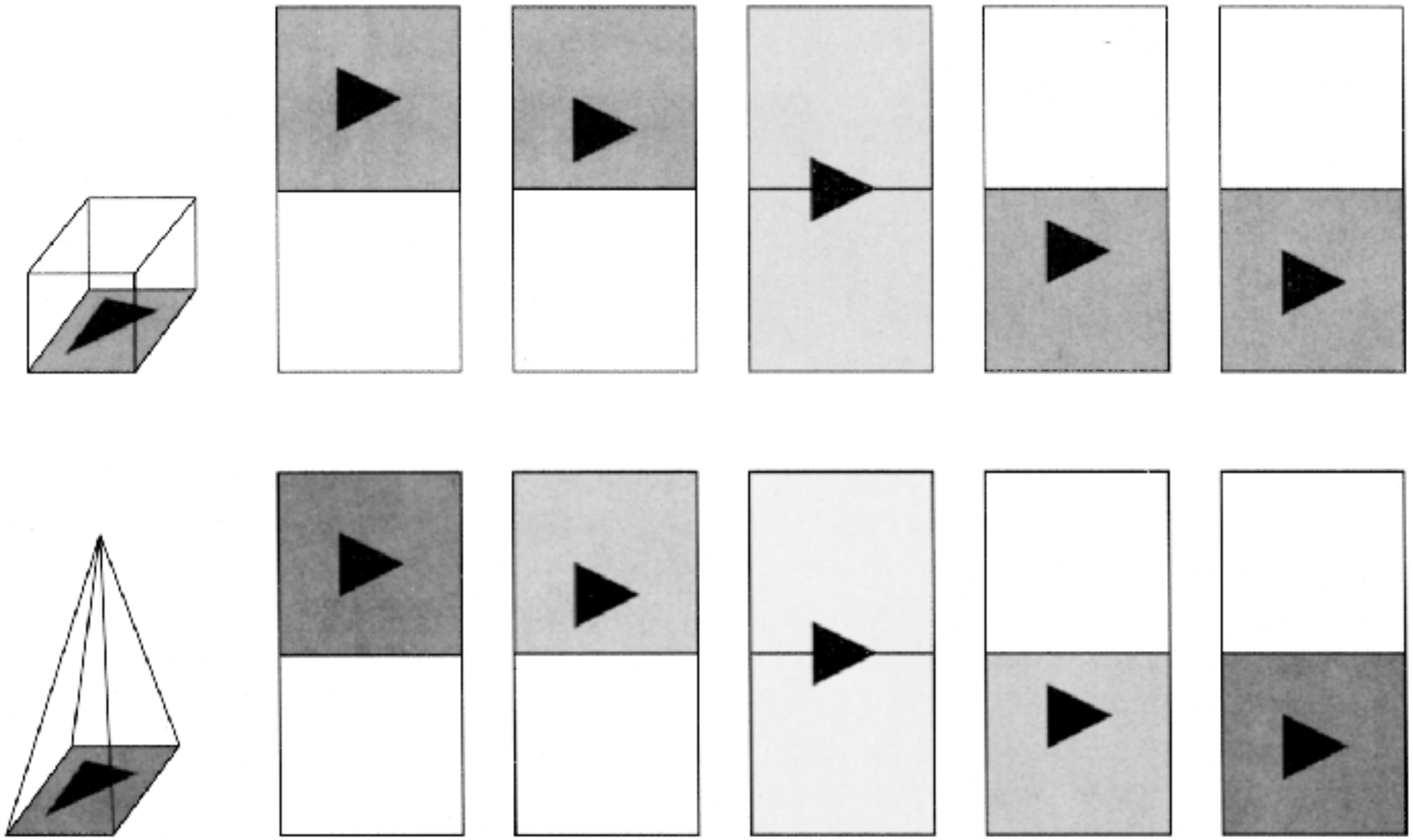


box filter cone Gaussian filter

Filter Kernels in Action



Filter Kernels in Action



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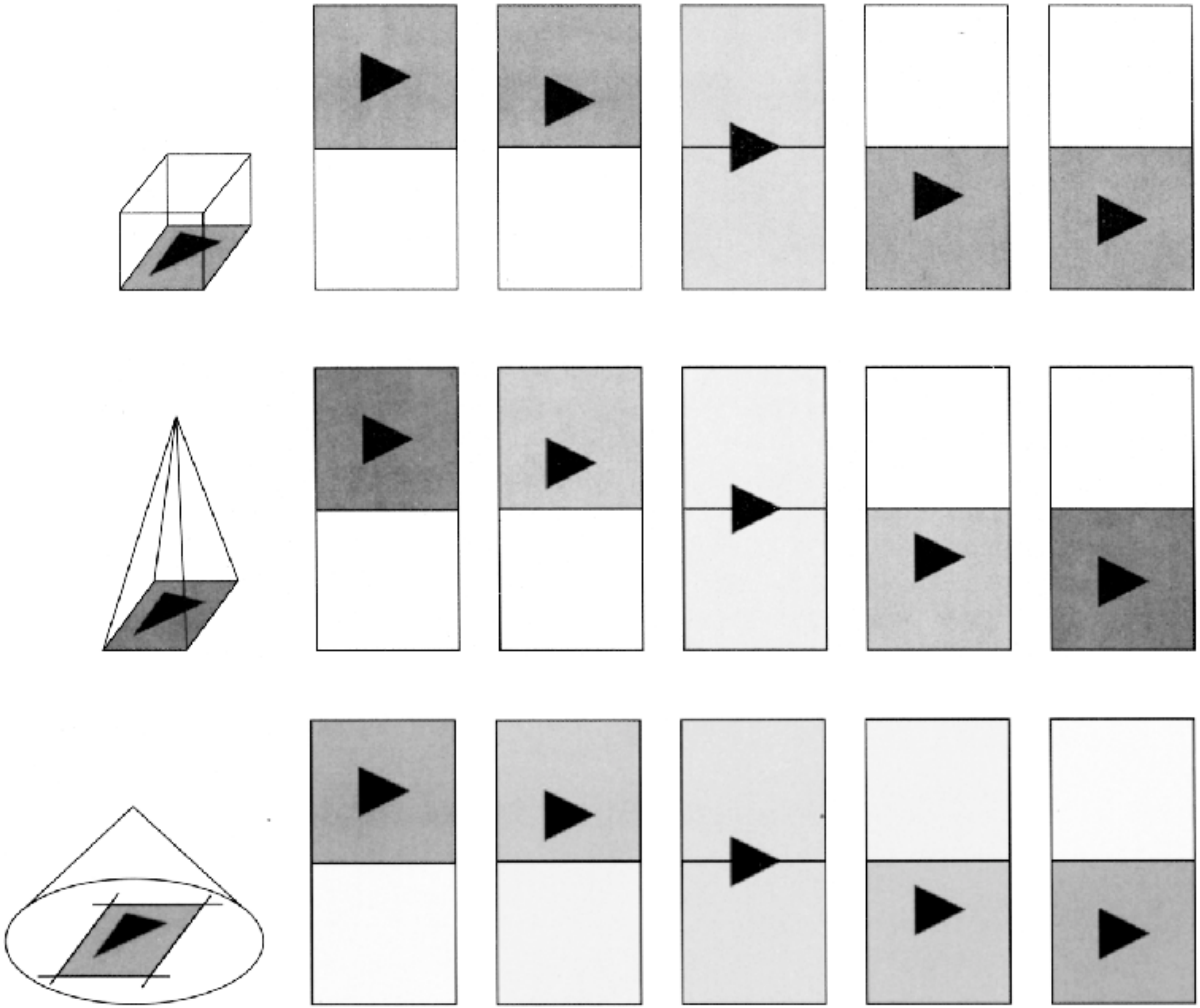

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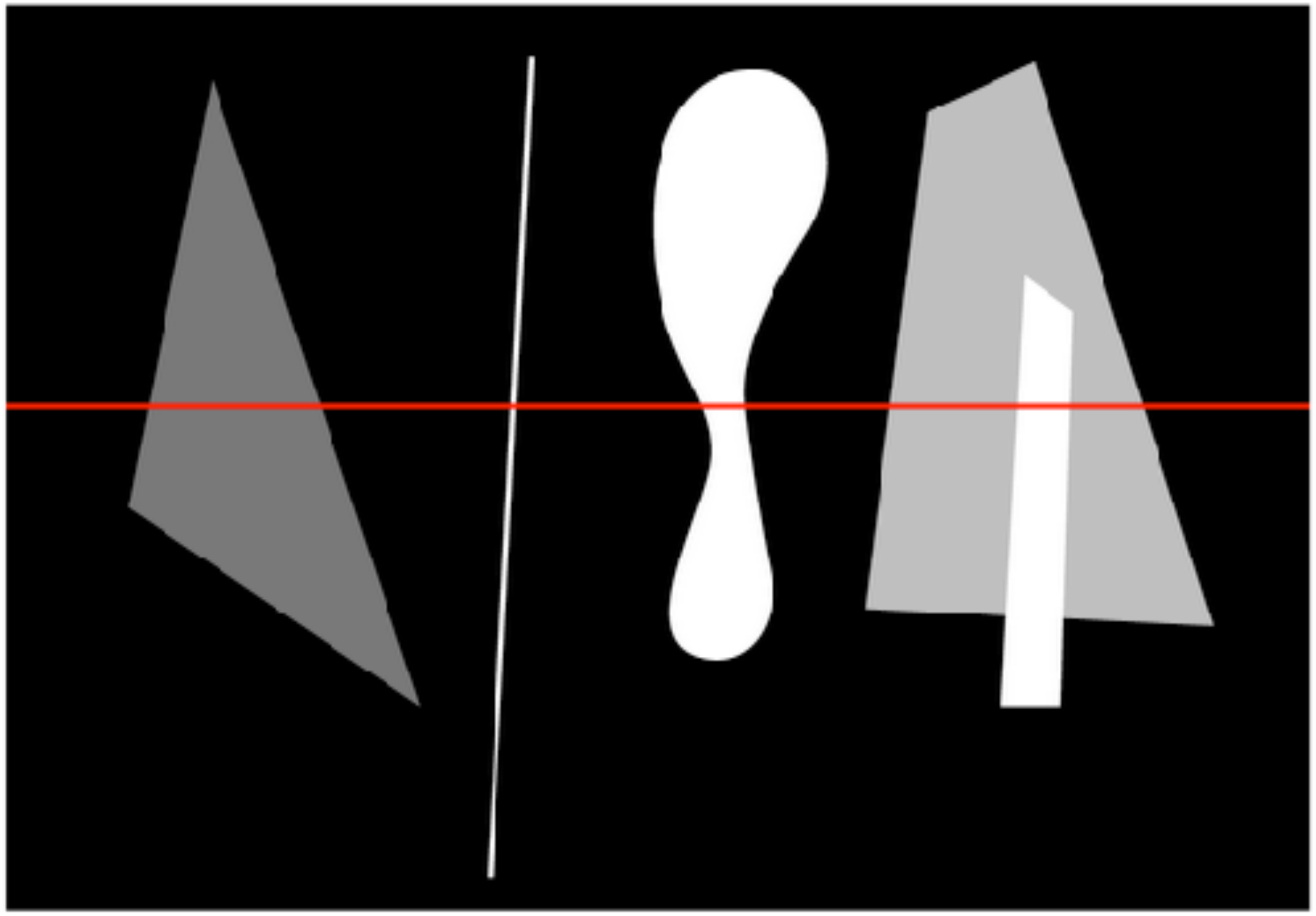
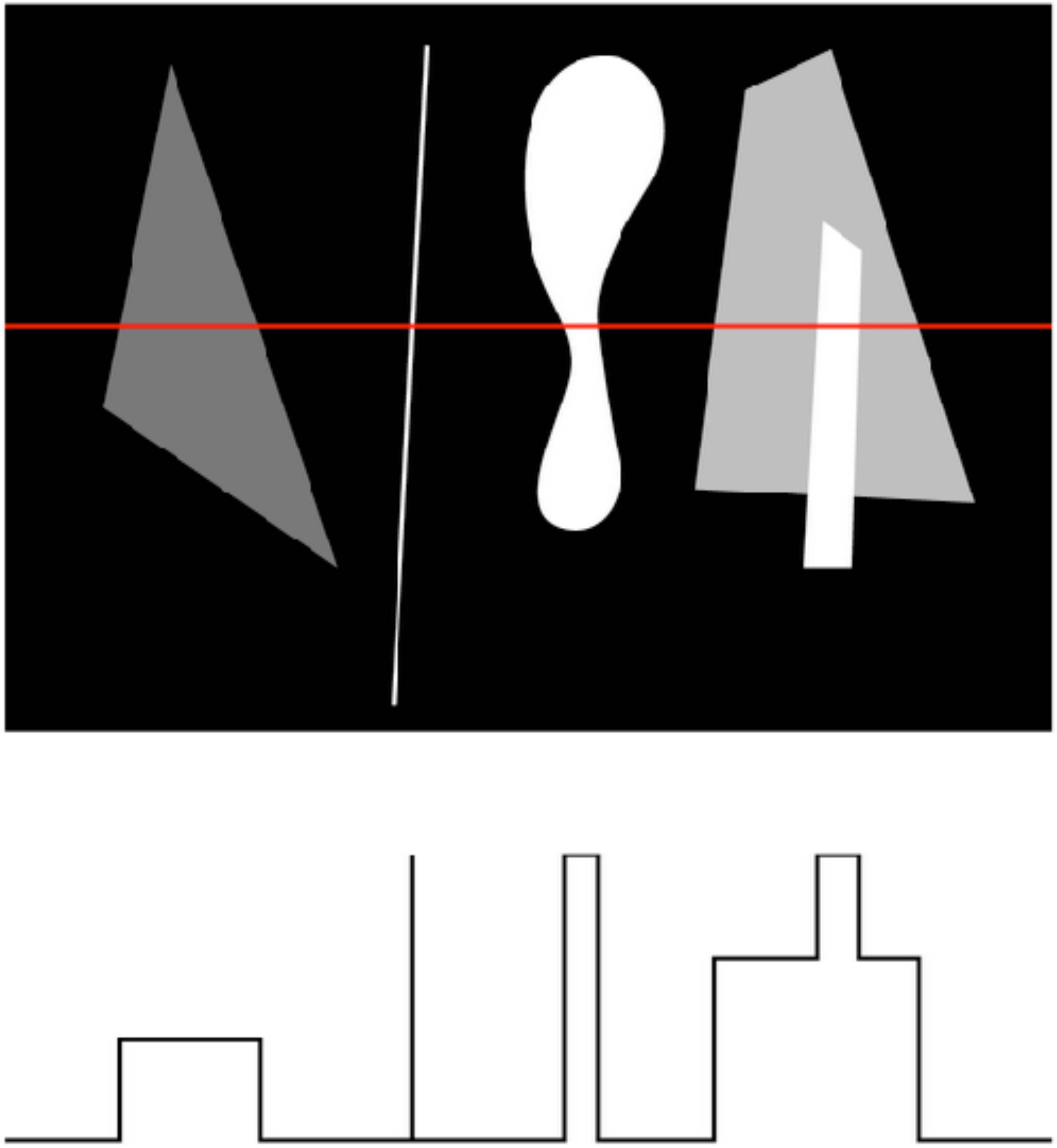
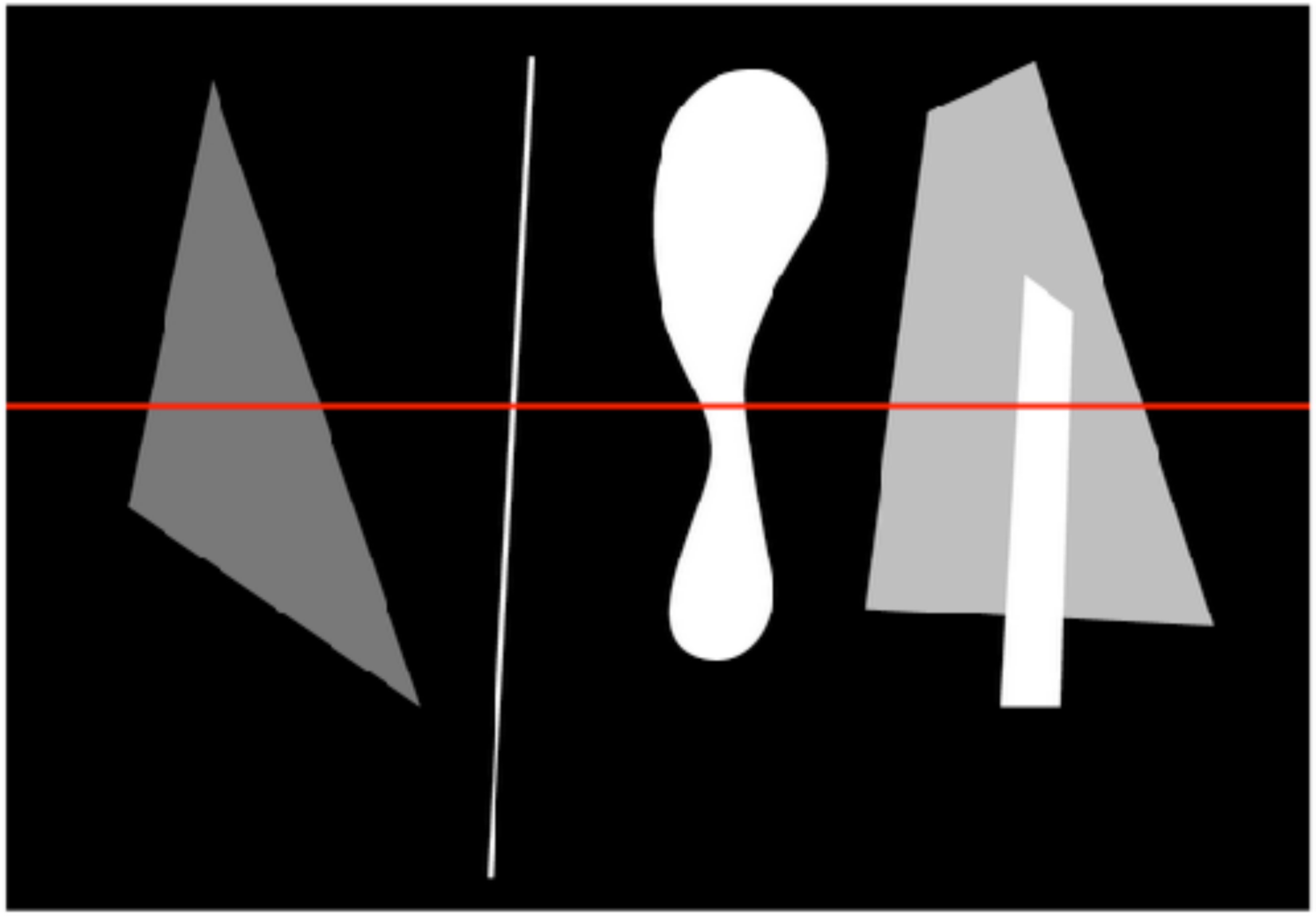
Scanline as Signal

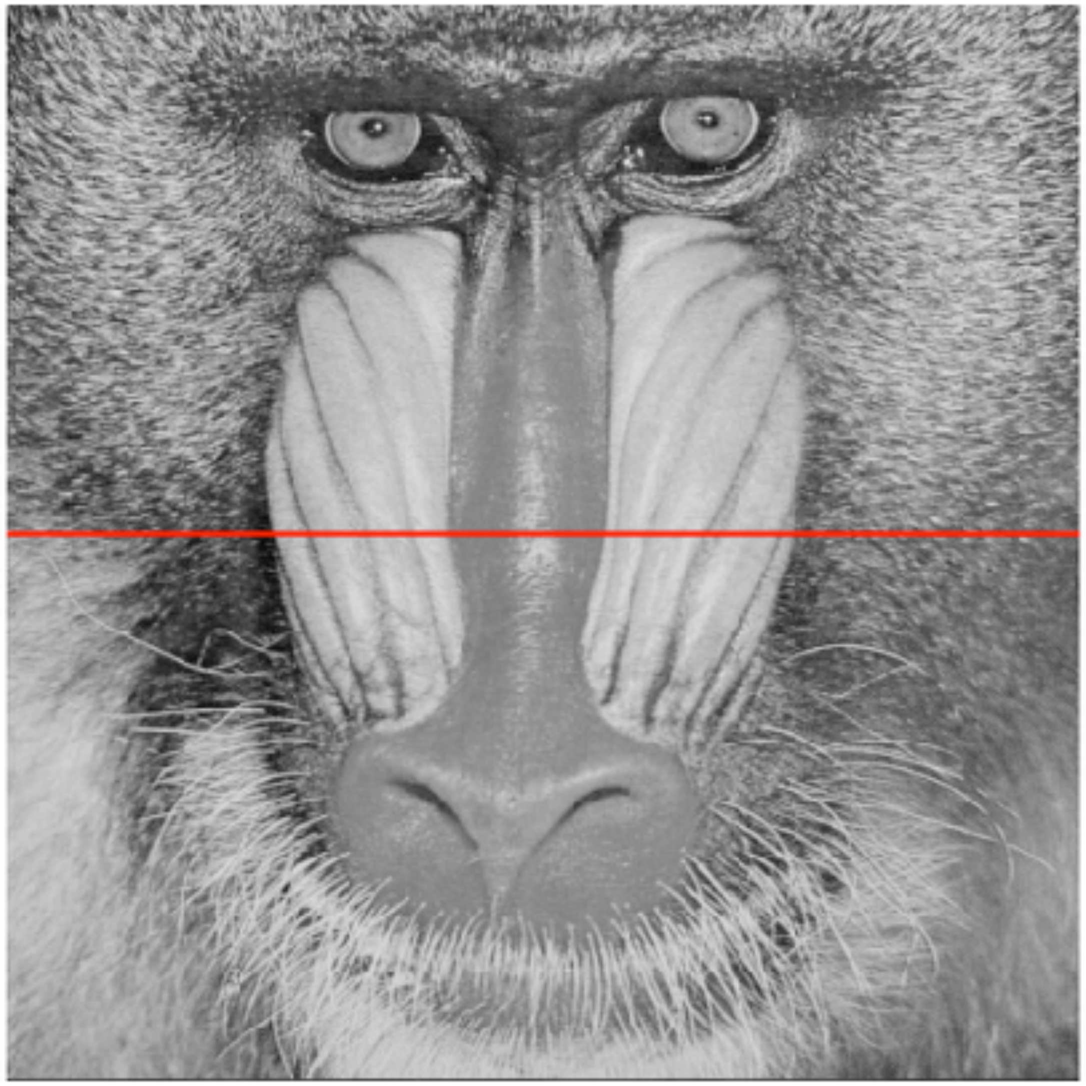


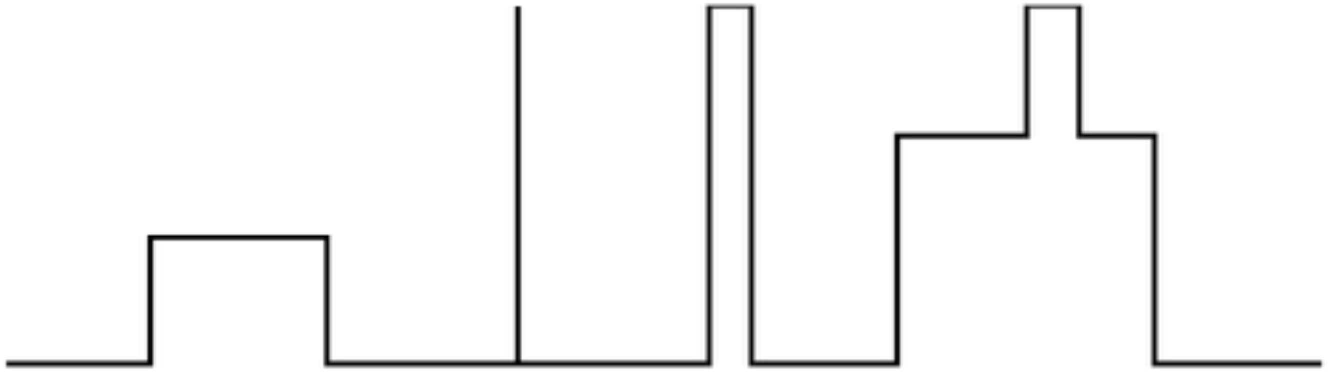
Scanline as Signal



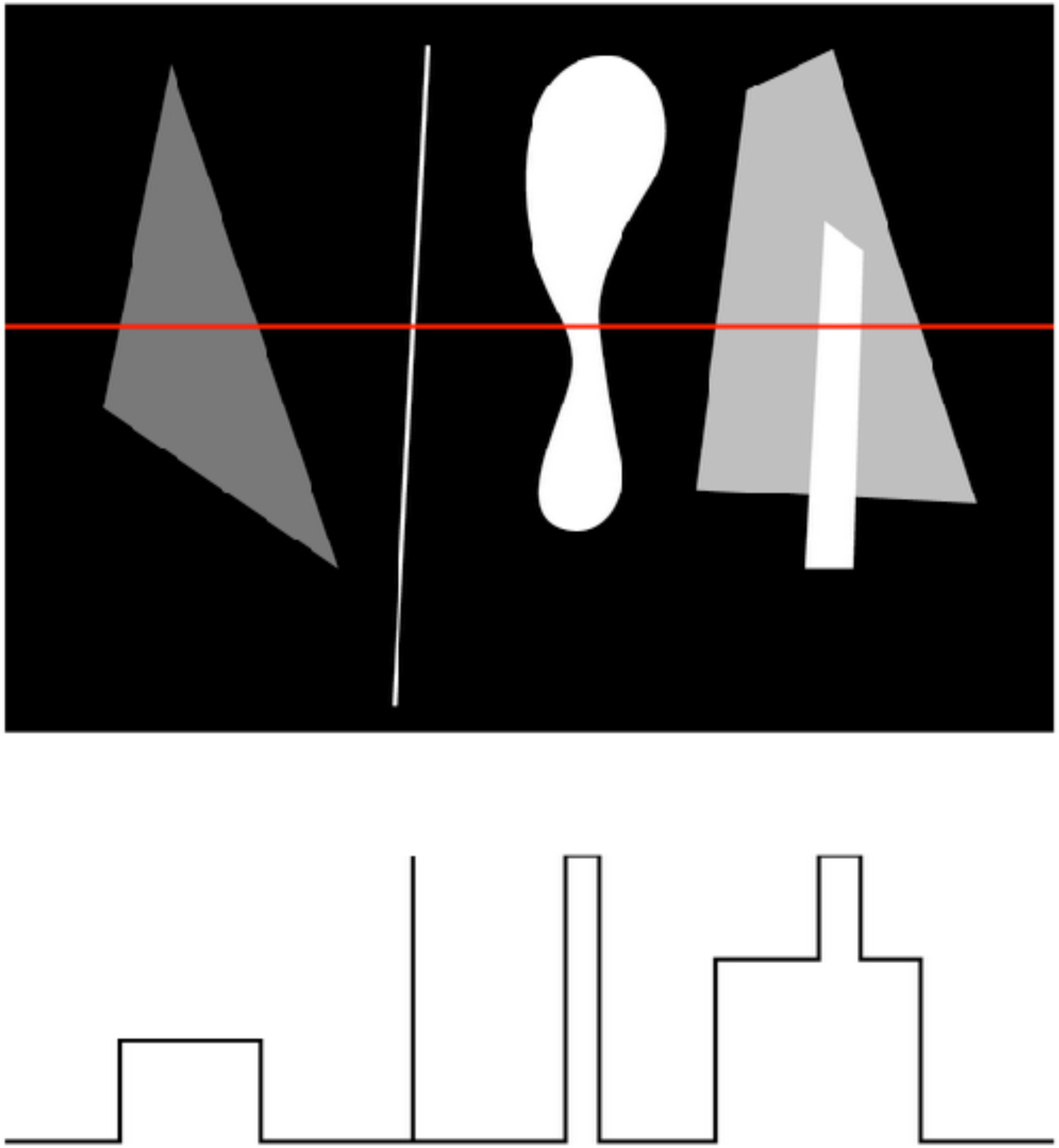
Scanline as Signal

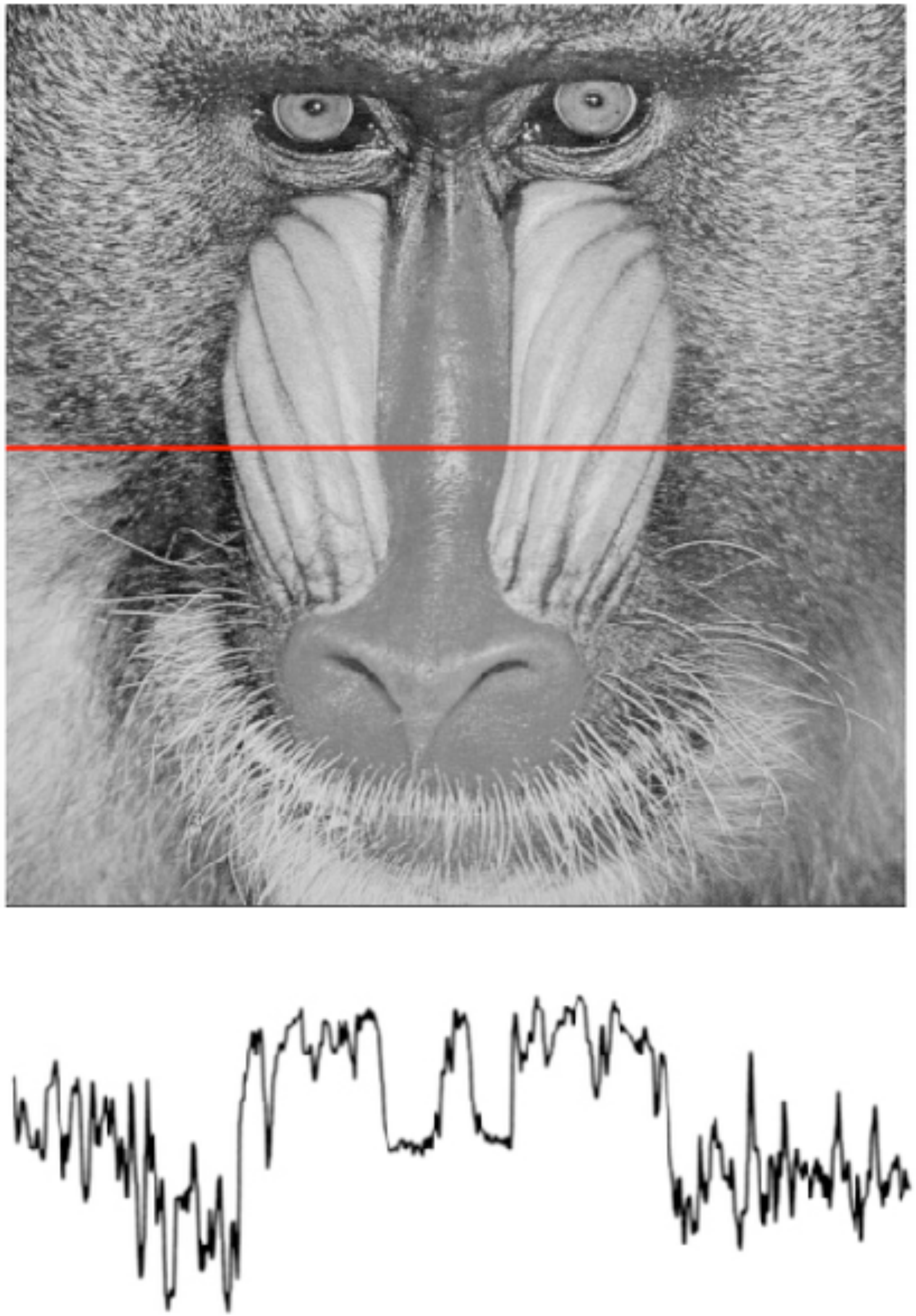




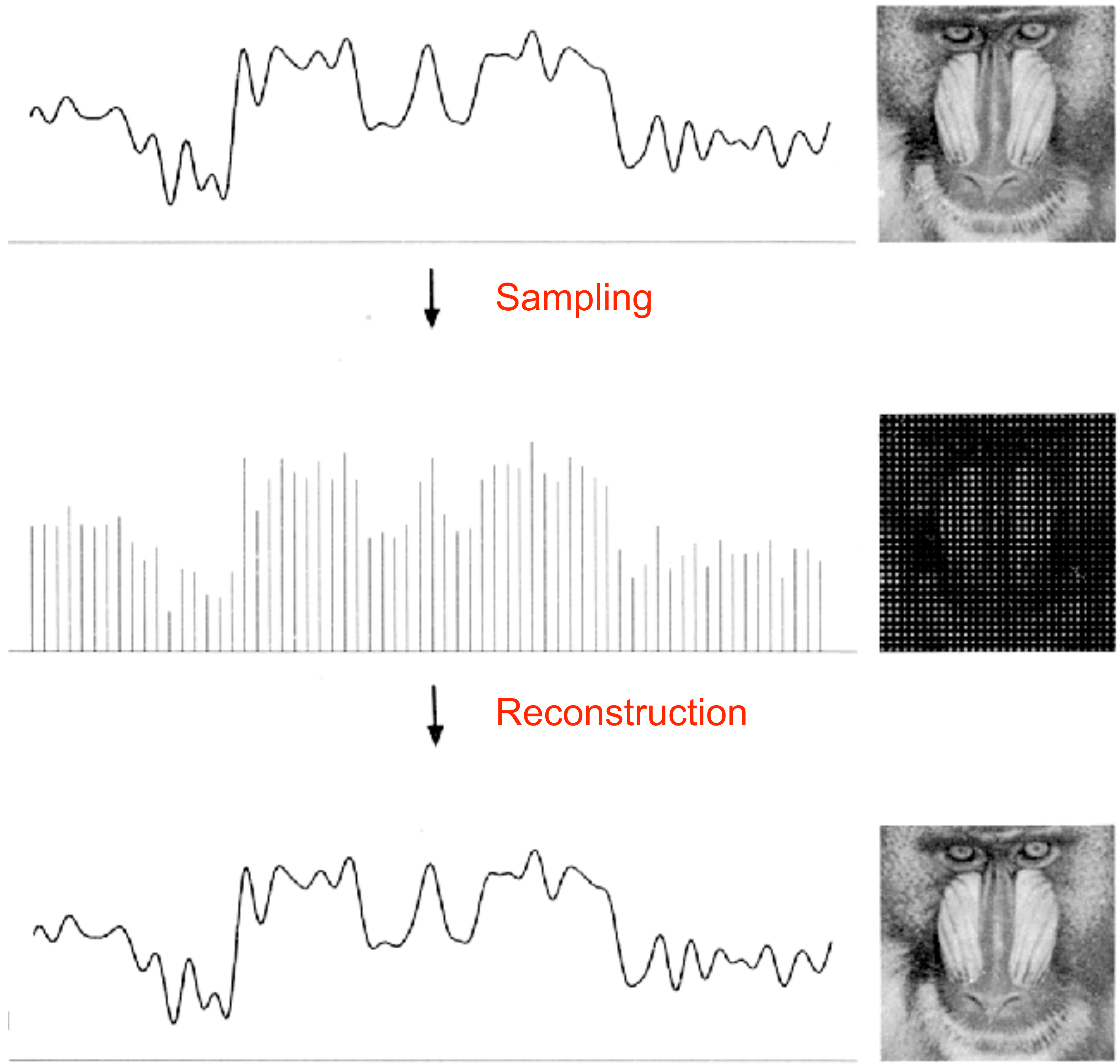


Scanline as Signal





Storing a Scanline



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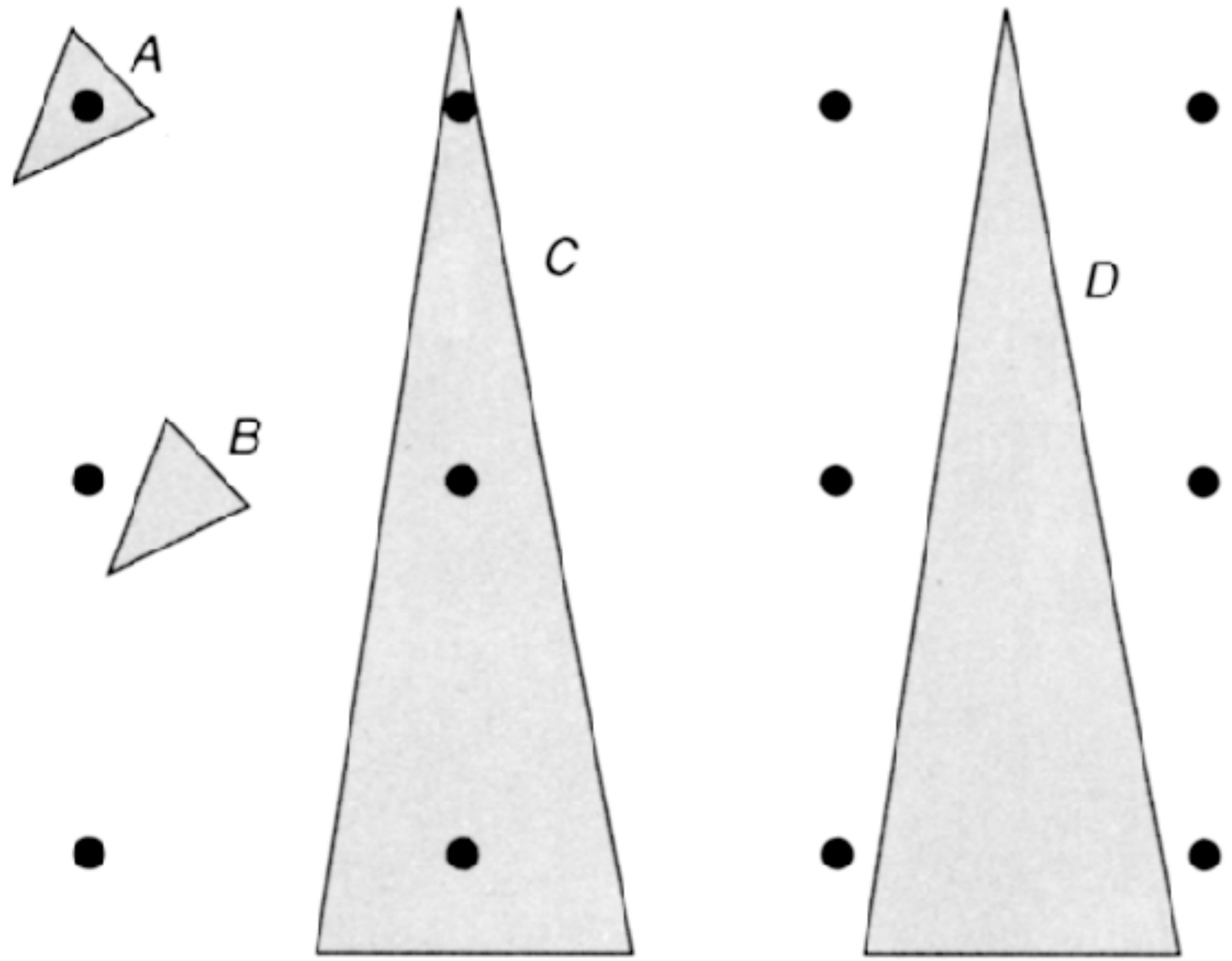
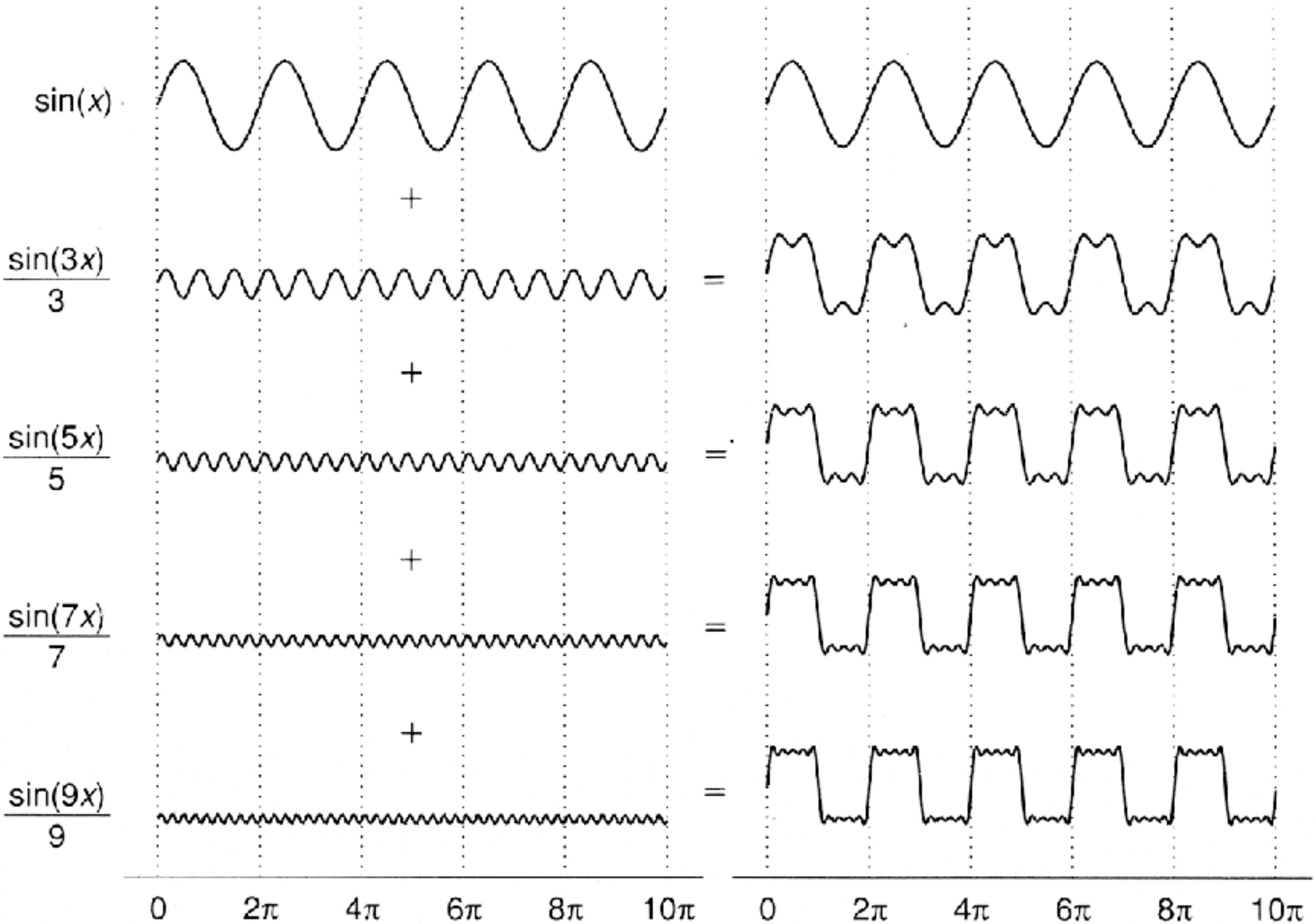
Sampling: Basic Problem

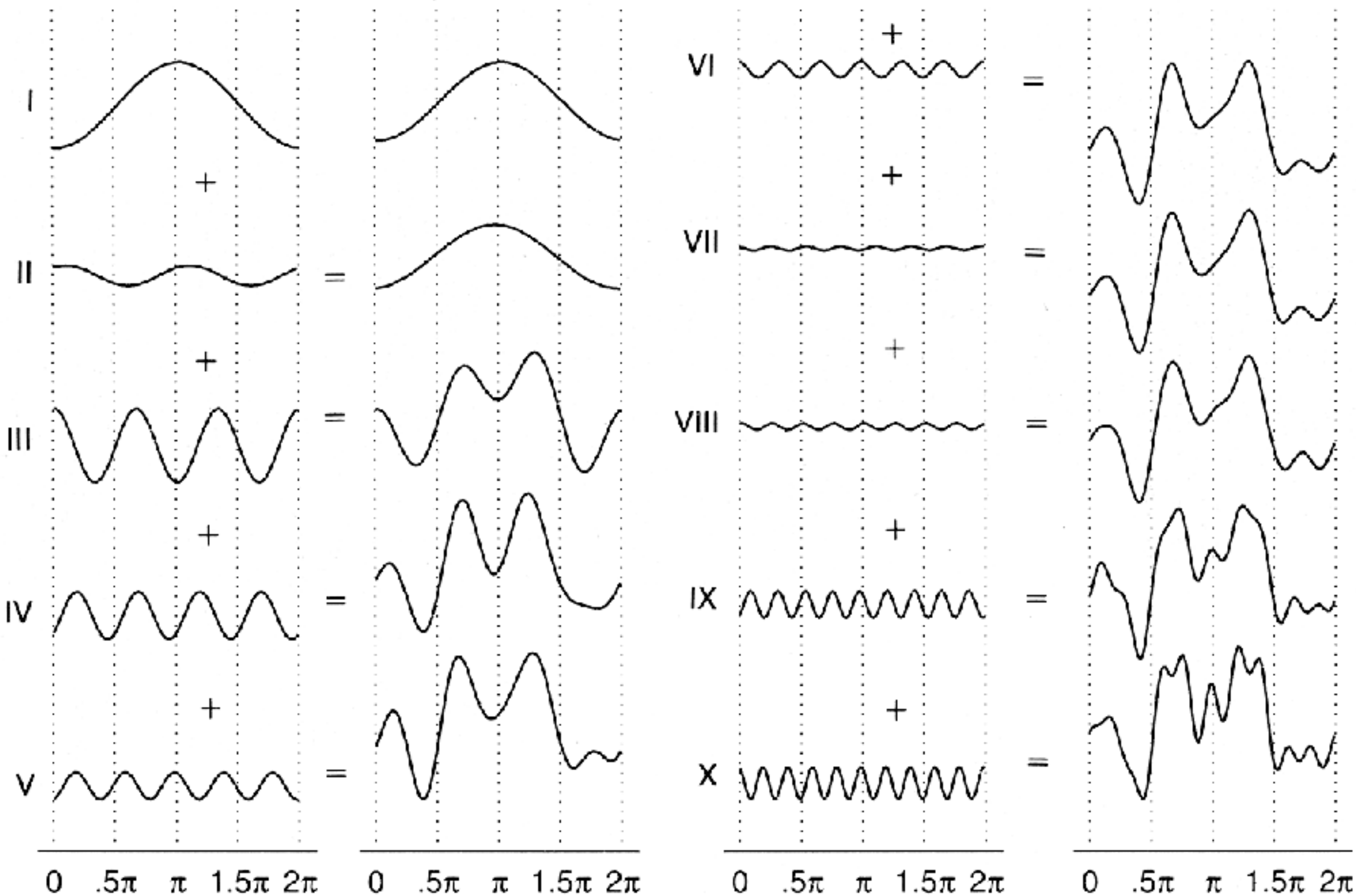


- **Without at least some knowledge about the sampled signal, we cannot guarantee that this will work!**
  - **At all!**

Square Wave Decomposition



Signal Decomposition



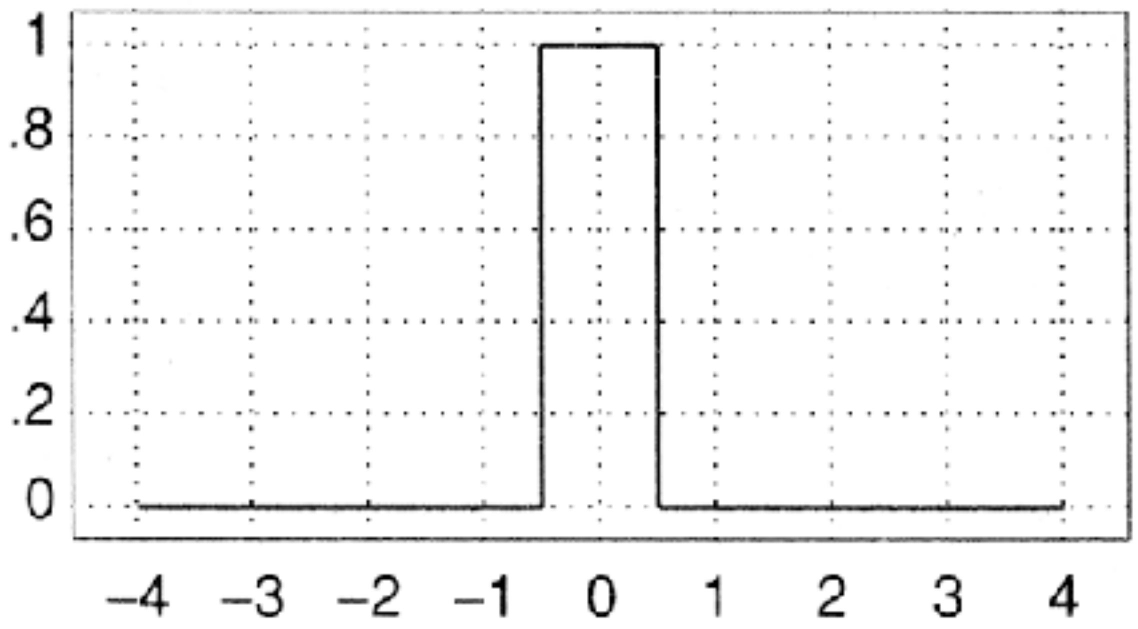
Fourier Transform

- **Connects time and frequency domain**
- **Applicable for arbitrary signals**
  - **f(x) time domain**
  - **F(u) frequency domain**

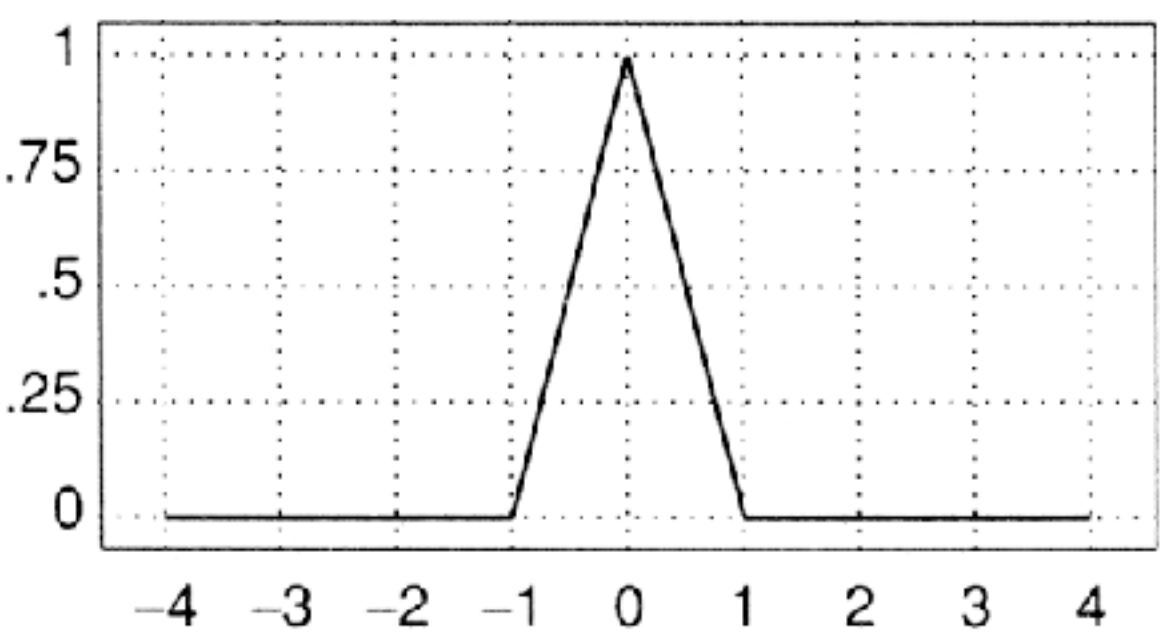
$$F(u) = \int_{-\infty}^{+\infty} f(x)[\cos 2\pi ux - i \sin 2\pi\, ux]\, dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(u)[\cos 2\pi ux - i \sin 2\pi\, ux]\, du$$

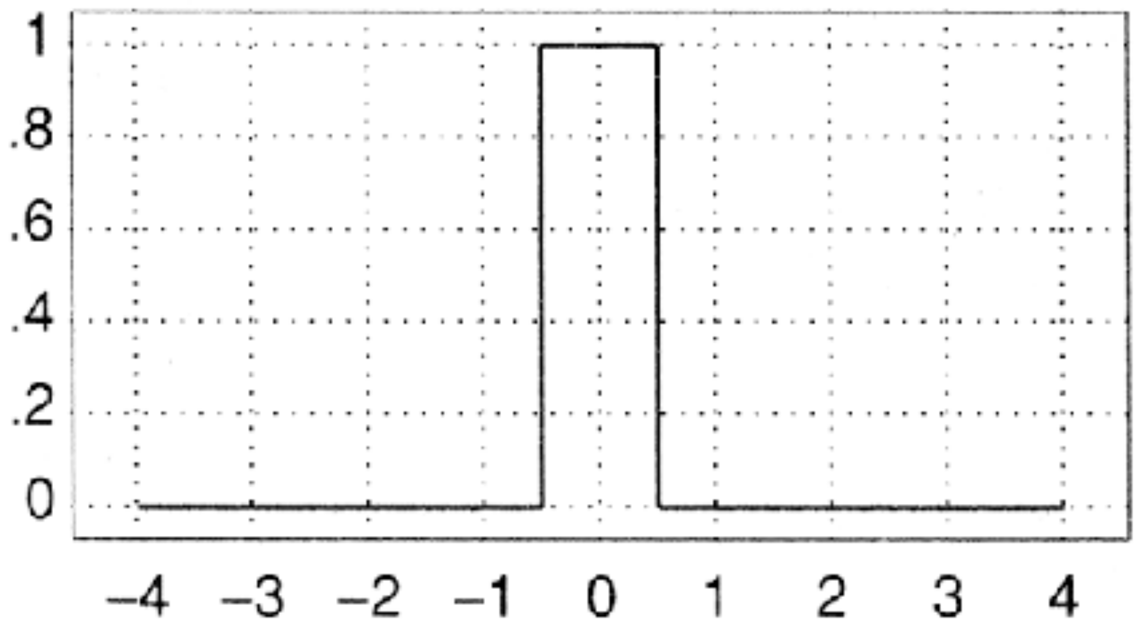
Box & Sawtooth



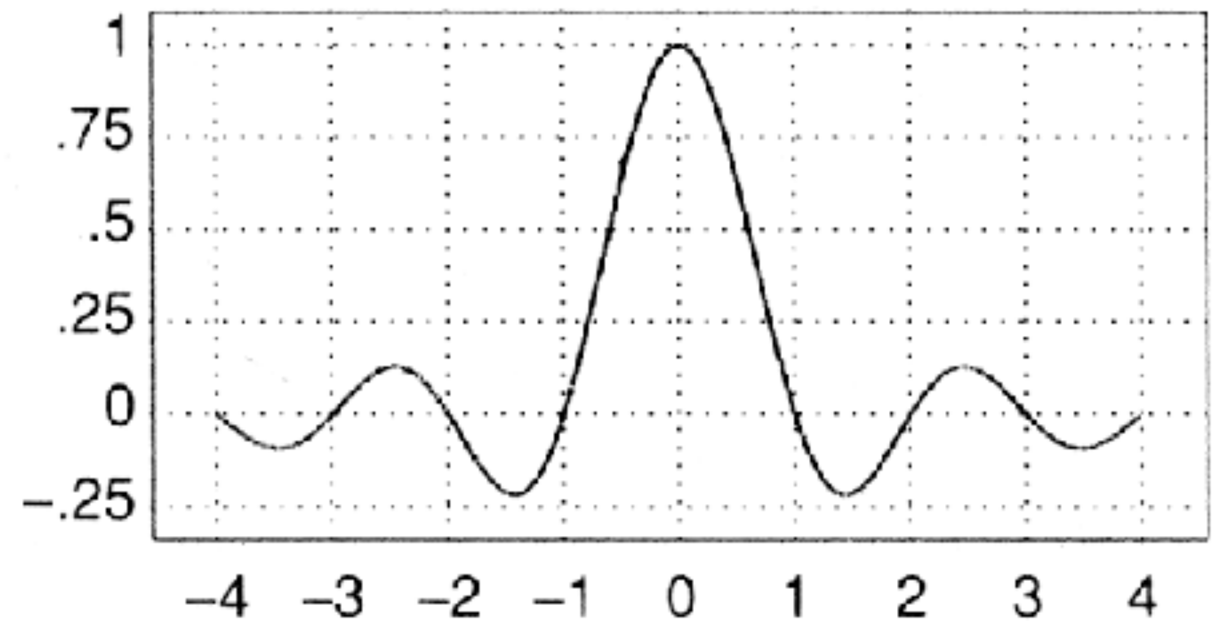
$f(x)$ $\quad$ $F(u)$



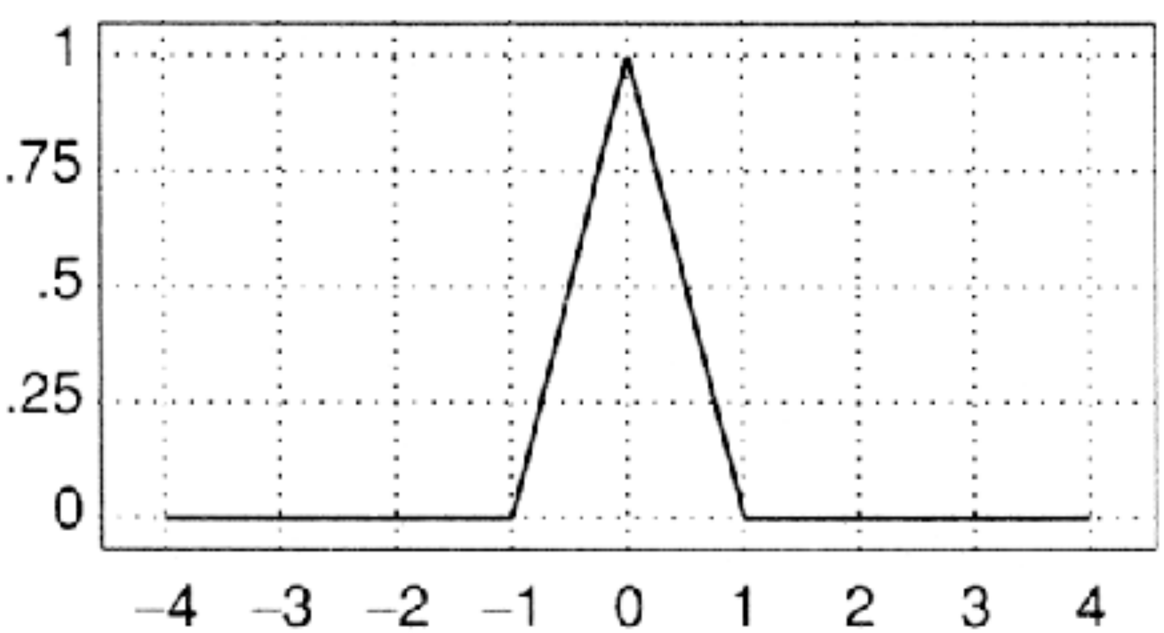
Box & Sawtooth



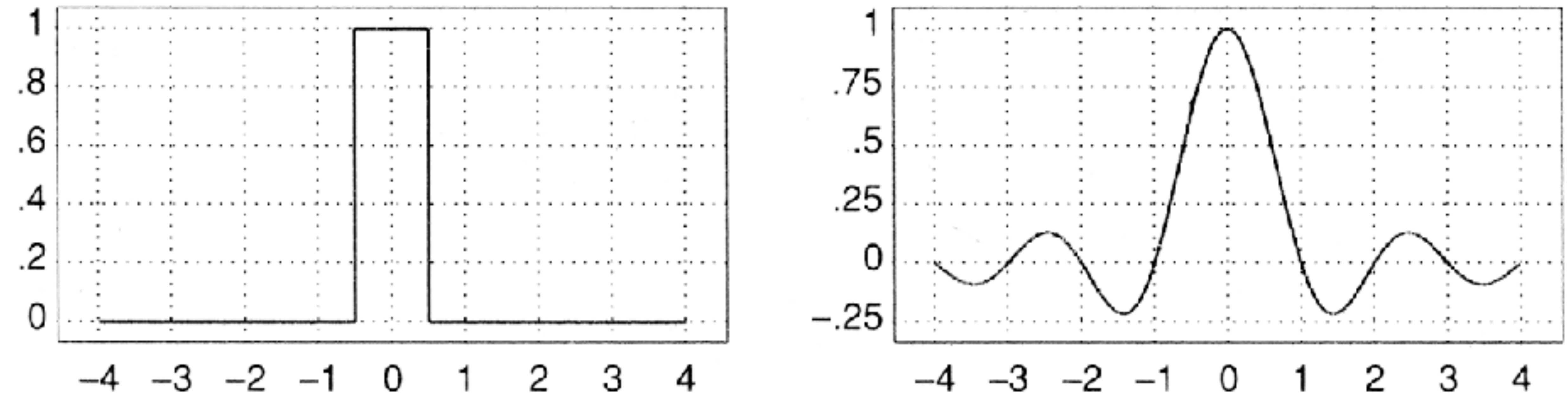
$f(x)$



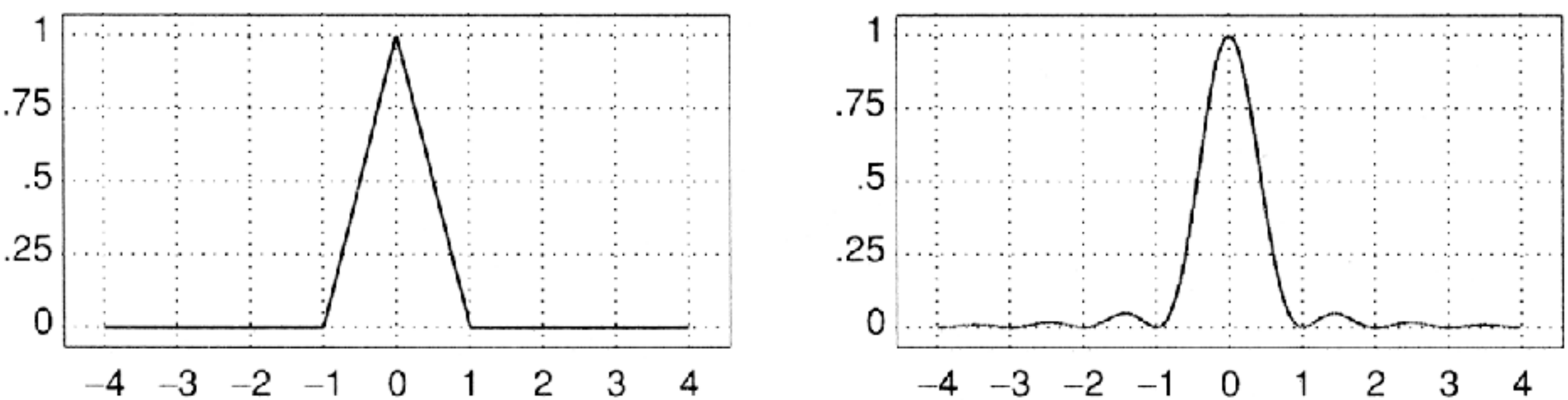
$F(u)$



Box & Sawtooth



$f(x)$ $F(u)$



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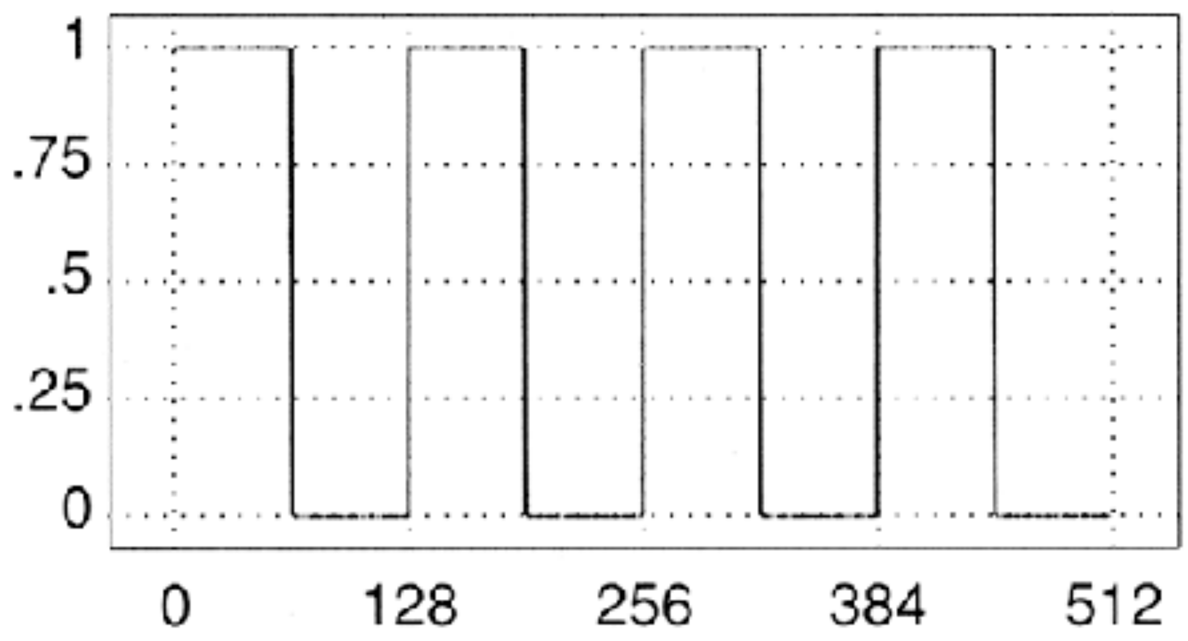
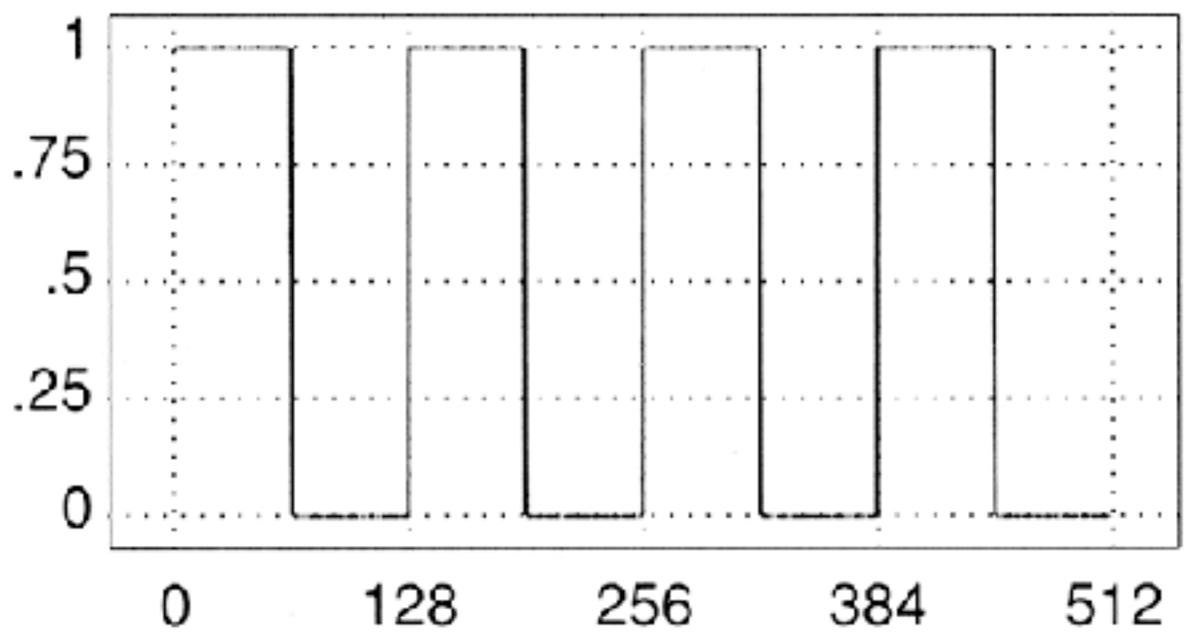
Square Wave & Scanline



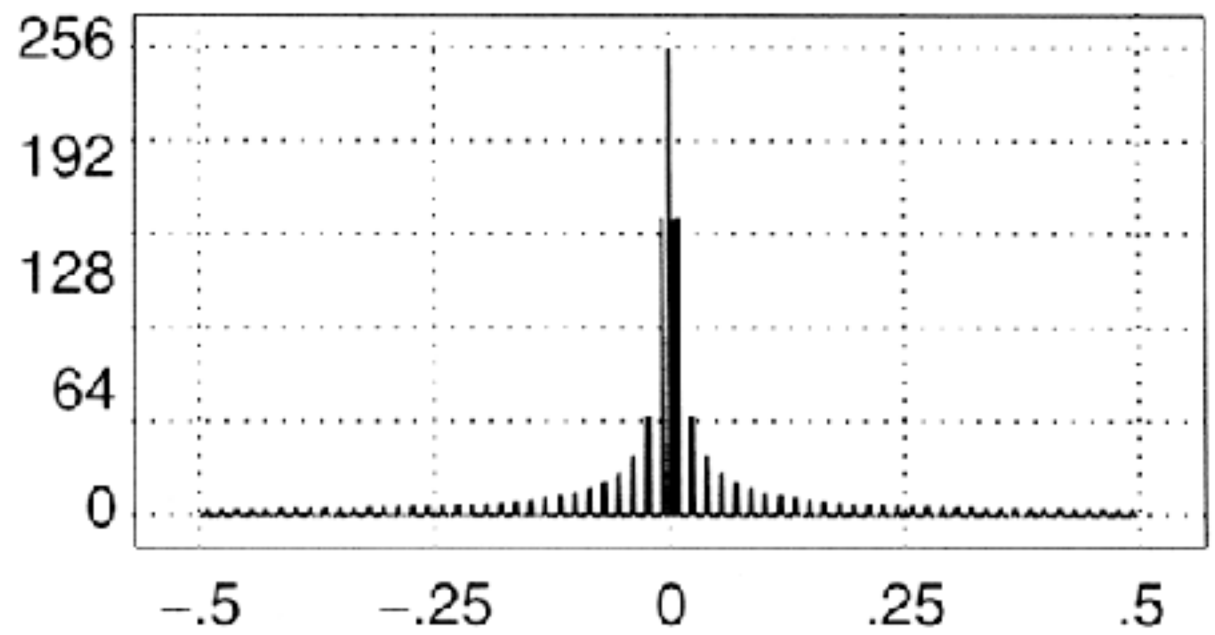
$f(x)$

$F(u)$

Square Wave & Scanline

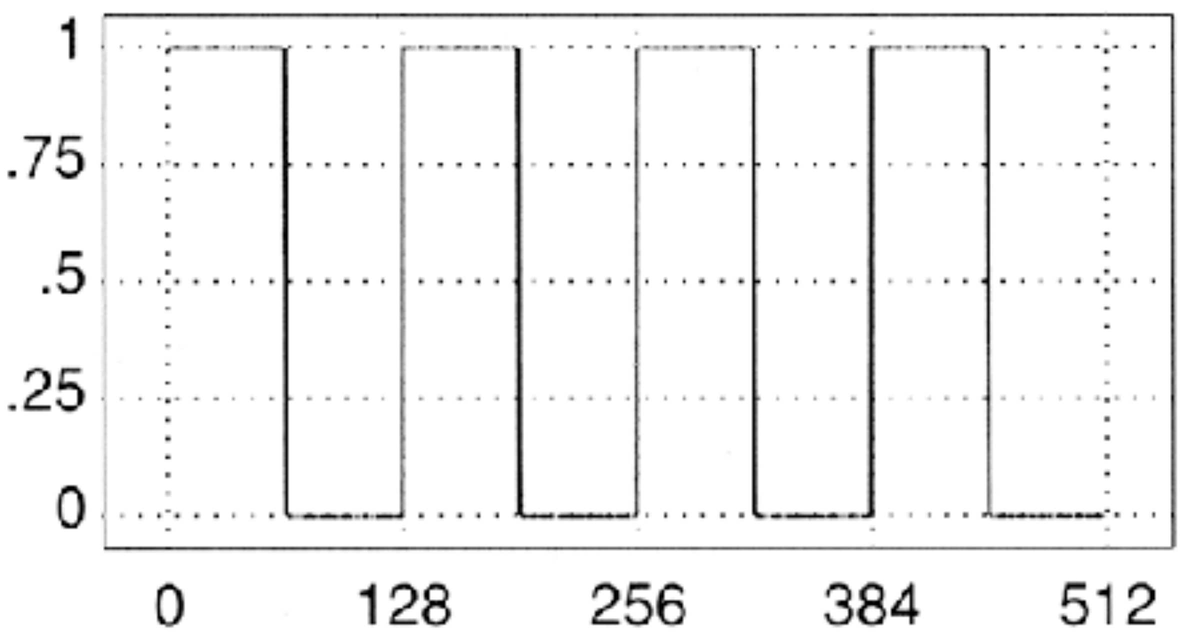


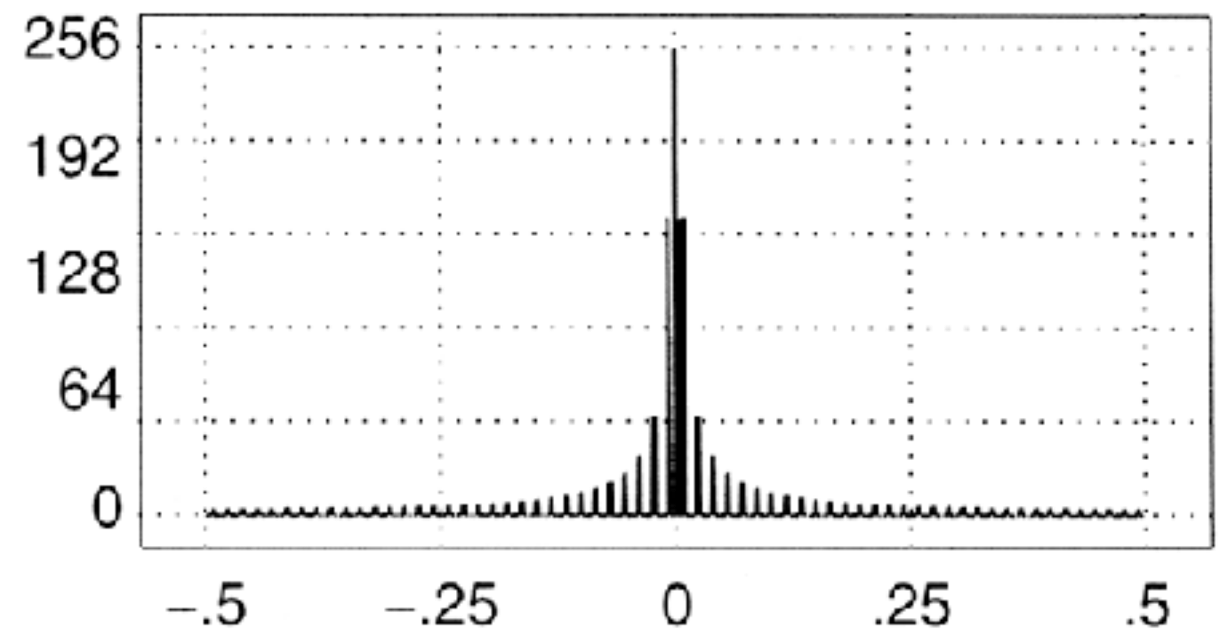
$f(x)$



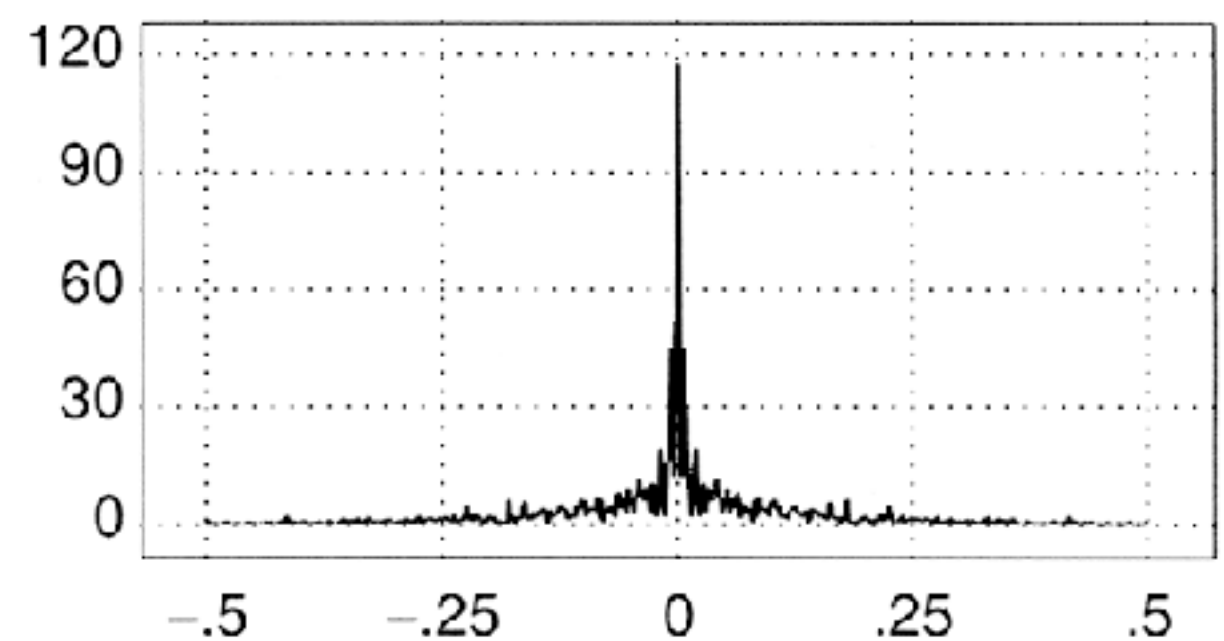
$F(u)$

Square Wave & Scanline





$f(x)$ $F(u)$



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Fourier Transform

- Yields complex-valued frequency space functions
  - The imaginary part contains phase information - this is usually omitted
- Can be generalised to higher dimensions (2D, 3D)
- Alternatives, such as the Hartley transform, exist

Discrete Fourier Transform

- **For discrete signals**

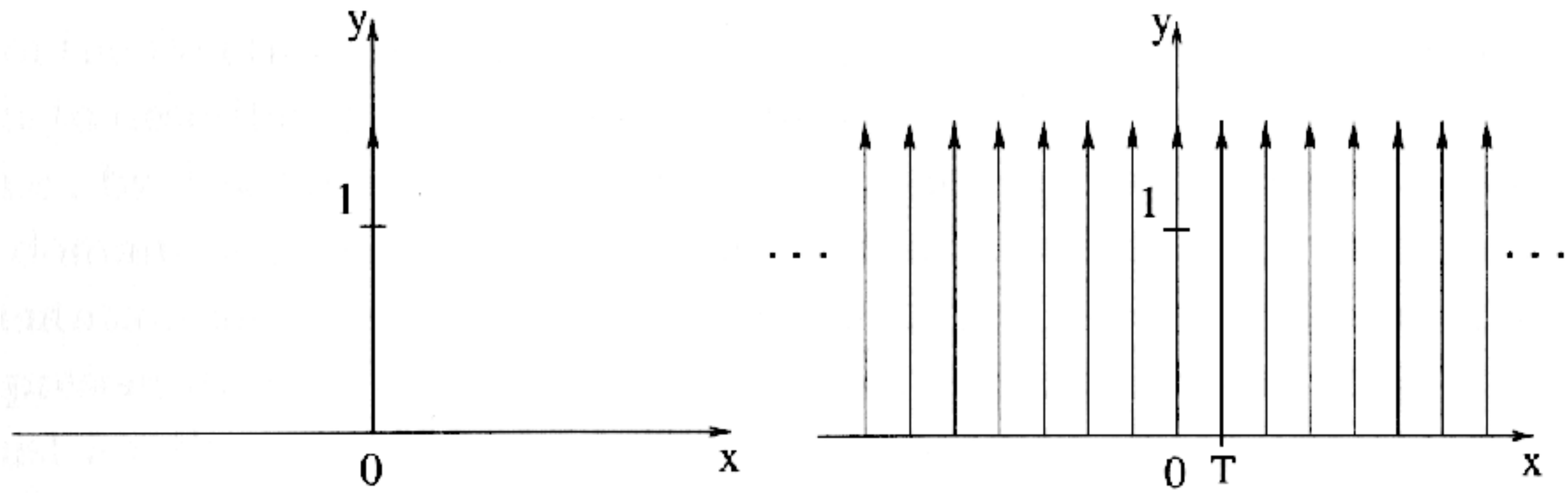
$$F(u) = \sum_{0}^{N-1} f(x)[\cos(2\pi ux/N) - i\sin(2\pi ux/N)], 0 \le u \le N-1$$

$$f(x) = \sum_{0}^{N-1} F(u)[\cos(2\pi ux/N) - i\sin(2\pi ux/N)], 0 \le x \le N-1$$

- **N samples: complexity O(N^2)**
- **Fast FT (FFT): O(N log N)**

Sampling Function: Comb

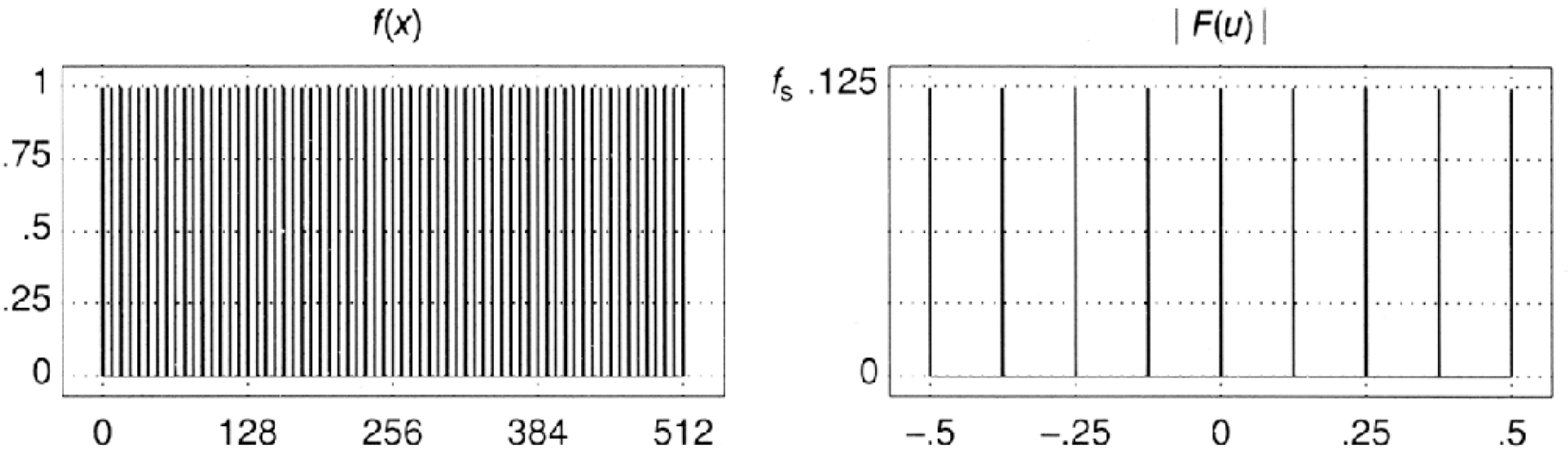
- A single Dirac pulse corresponds to sampling an image at a single location
- Regular sampling can be seen as multiplication with n evenly spaced Dirac pulses (comb function)



FT of a Comb

- **A time domain comb corresponds to a frequency domain comb with inverse pulse distance**

$$\mathrm{comb}_T(x) \equiv \mathrm{comb}_{1/T}(\omega)$$



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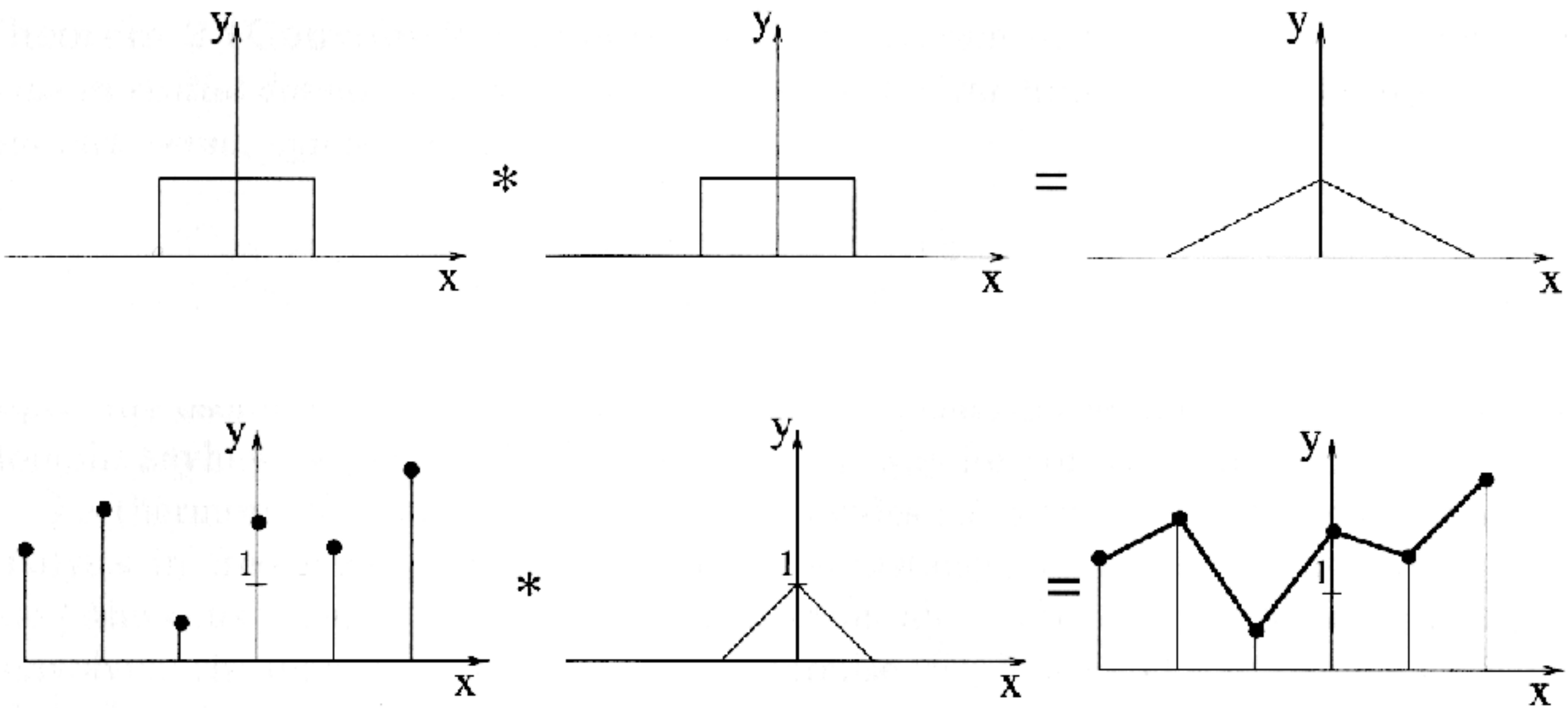
Convolution

- Mathematical operator:
  - Two functions as input
  - A new function as output
- “Weighing the first function with the second”

$$(f * g)(t) = \int_D f(\tau)g(t-\tau)d\tau$$

Convolution Examples



Convolution Theorem

- Convolution in the time domain corresponds to point wise multiplication in the frequency domain
- And vice versa

$$f * g \equiv F \cdot G$$

$$f \cdot g \equiv F * G$$

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Low Pass Filter

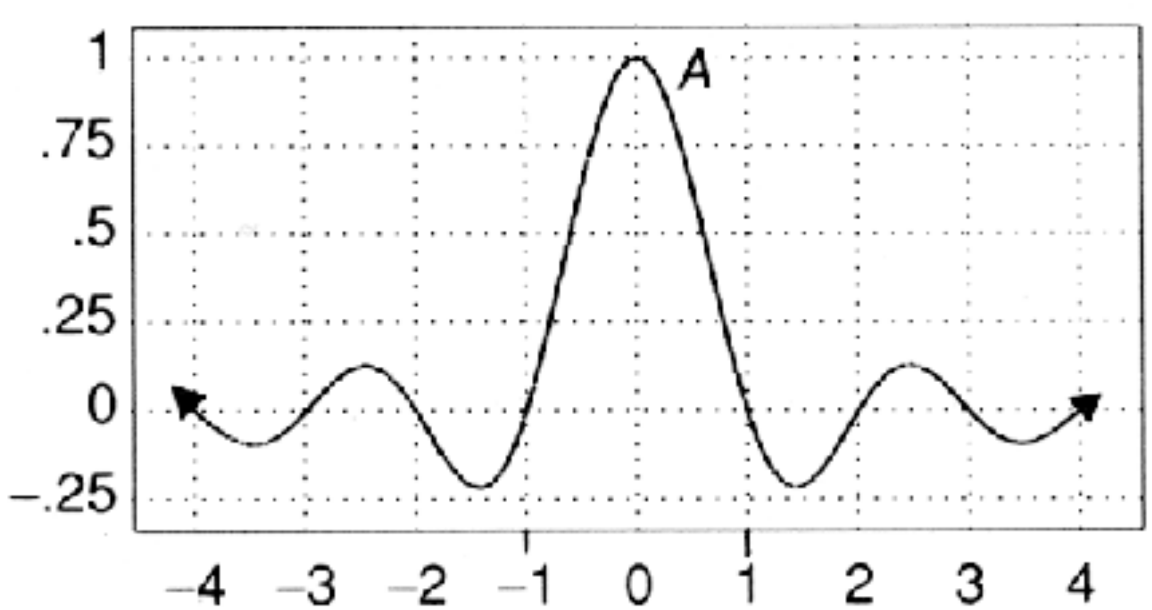
- Goal: low pass filtering of a scanline
- Time domain: convolution with a Sinc()
- Frequency domain: cutting off high frequencies - multiplication with a box function

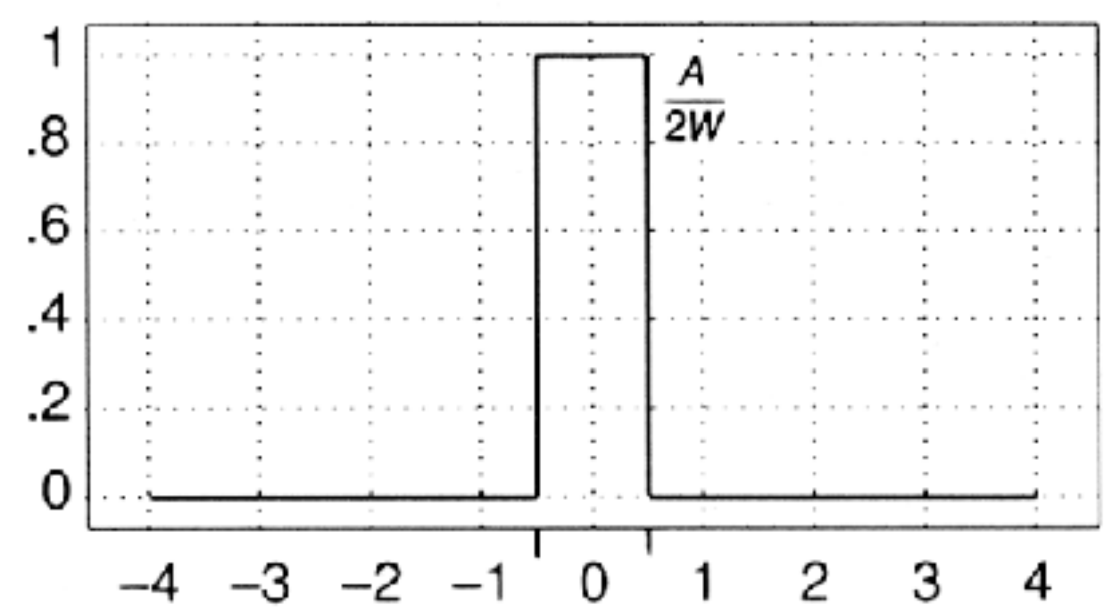
- Sinc() in the time domain corresponds to a box in the frequency domain

Sinc()

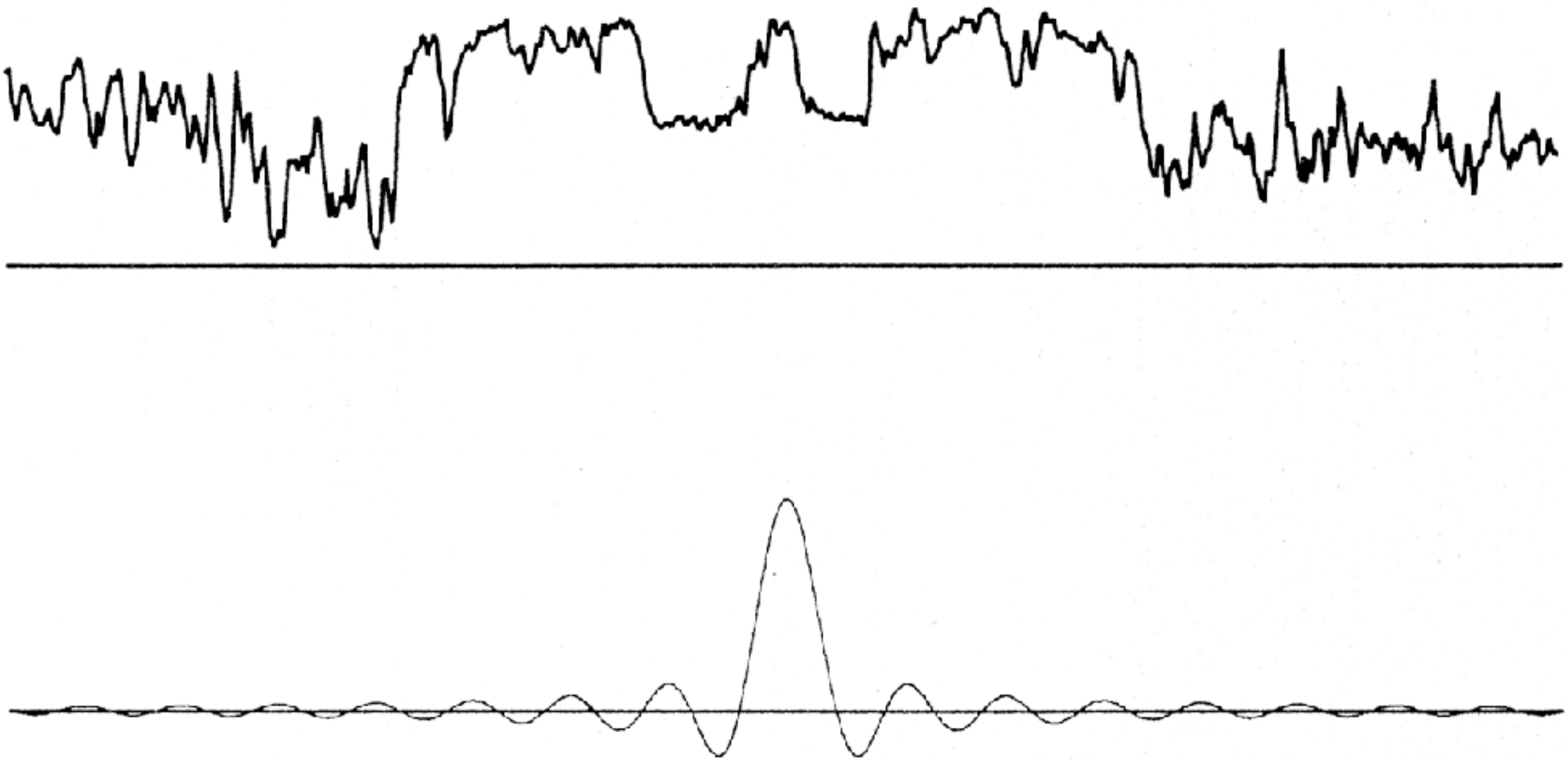
- Unlimited domain
- Perfect reconstruction filter
- Fourier transform of a box function

$$\mathrm{Sinc}(x) = \begin{cases} \dfrac{\mathrm{Sin}(x)}{x} & \mathit{falls}\ x \neq 0 \\ 0 & \mathit{falls}\ x = 0 \end{cases}$$

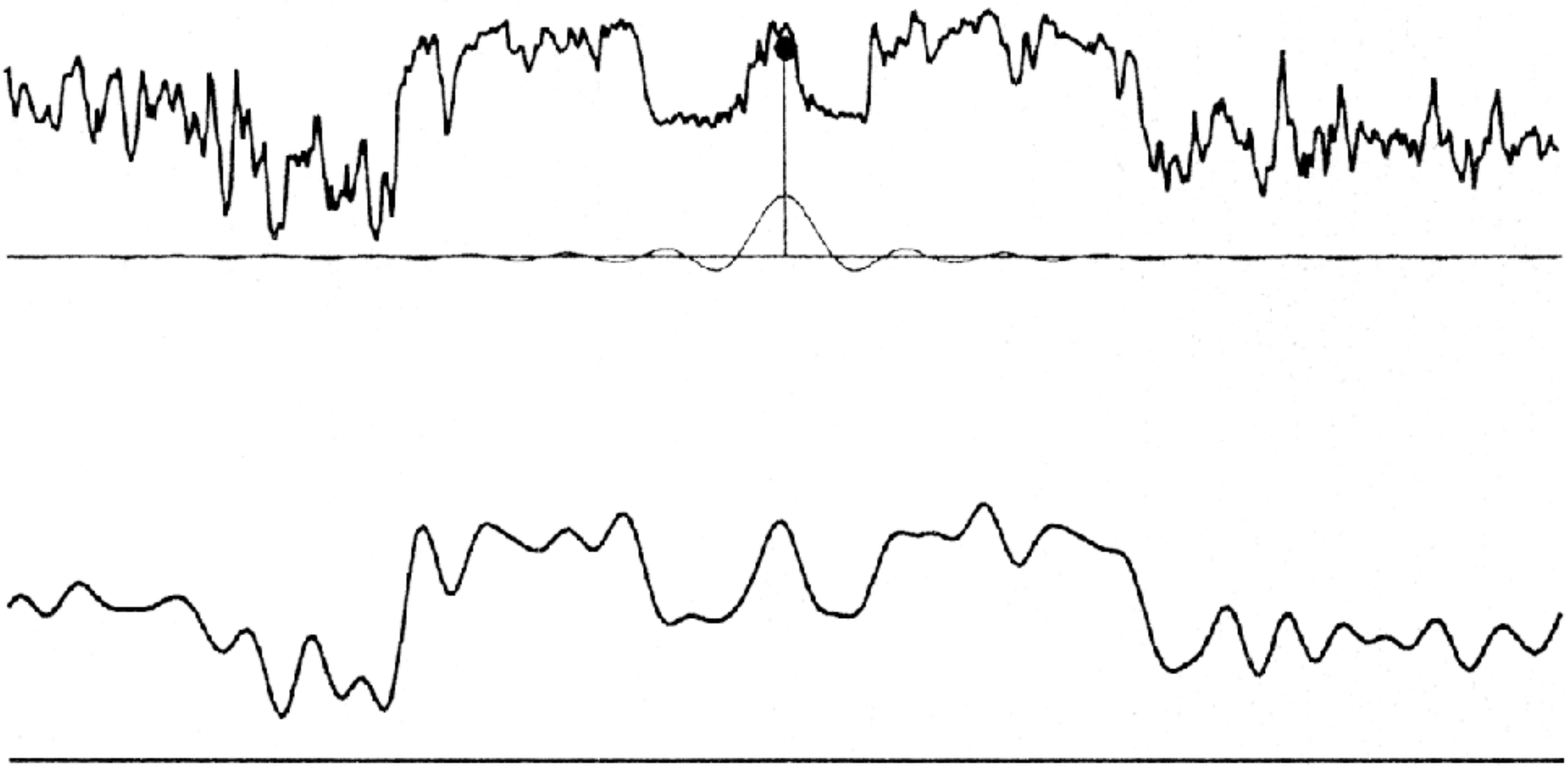




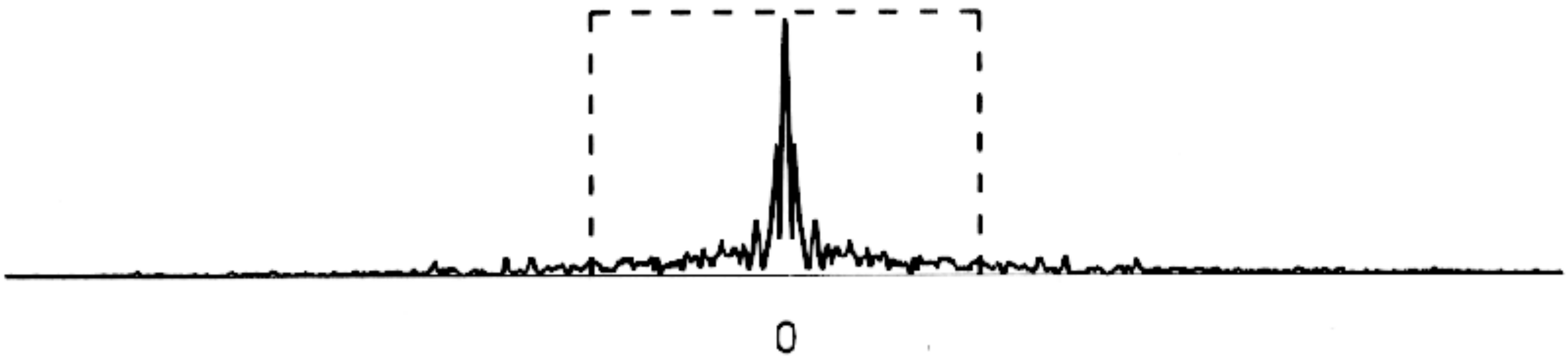
Low Pass (time domain)



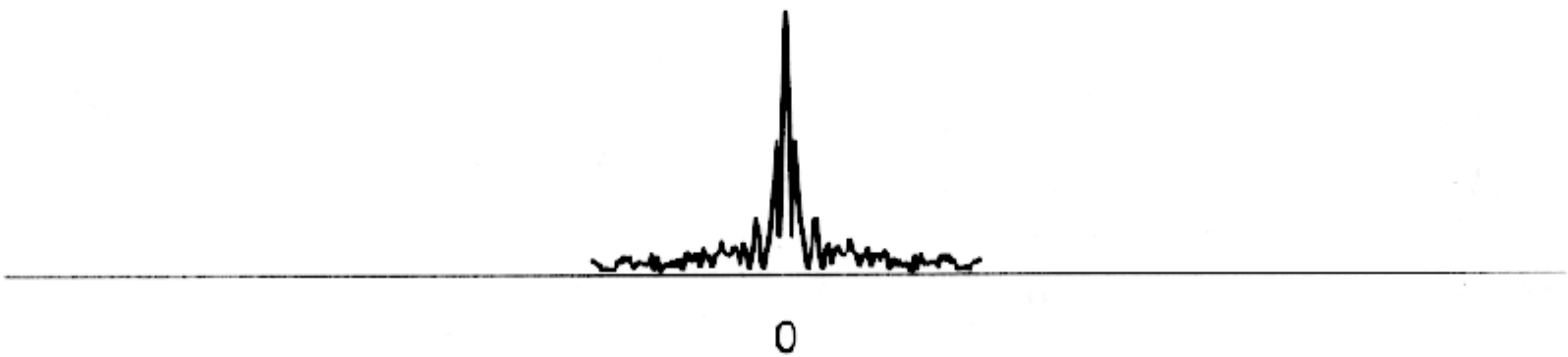
Low Pass (time domain)



Low Pass (freqency domain)



(c)



(d)

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Sampling

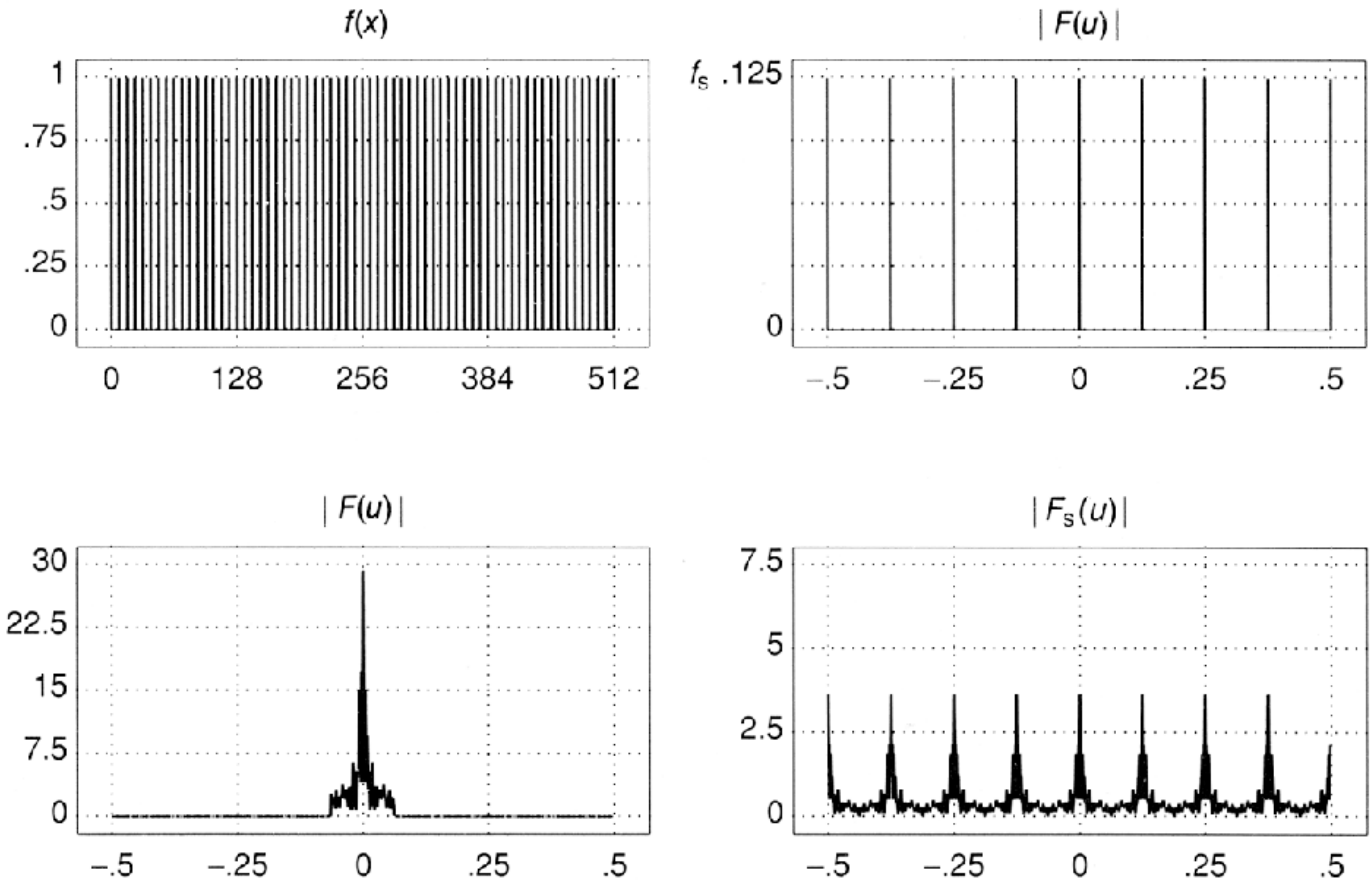
- **Sampling is a multiplication of the source signal with a comb function:**

$$f_s(x) = f(x) \cdot \mathrm{comb}_T(x)$$

- **In the frequency domain, this corresponds to a convolution (!) with an inversely spaced comb:**

$$F_s(\omega) = F(\omega) * \mathrm{comb}_{1/T}(\omega)$$

Spectrum of a Sampled Function



Perfect Reconstruction

- The copies of the original frequency spectrum must not overlap in the frequency domain
- Multiplication of the spectrum with a box function is equivalent to „cutting out" of the original spectrum
- This corresponds to a convolution with a Sinc function in the time domain

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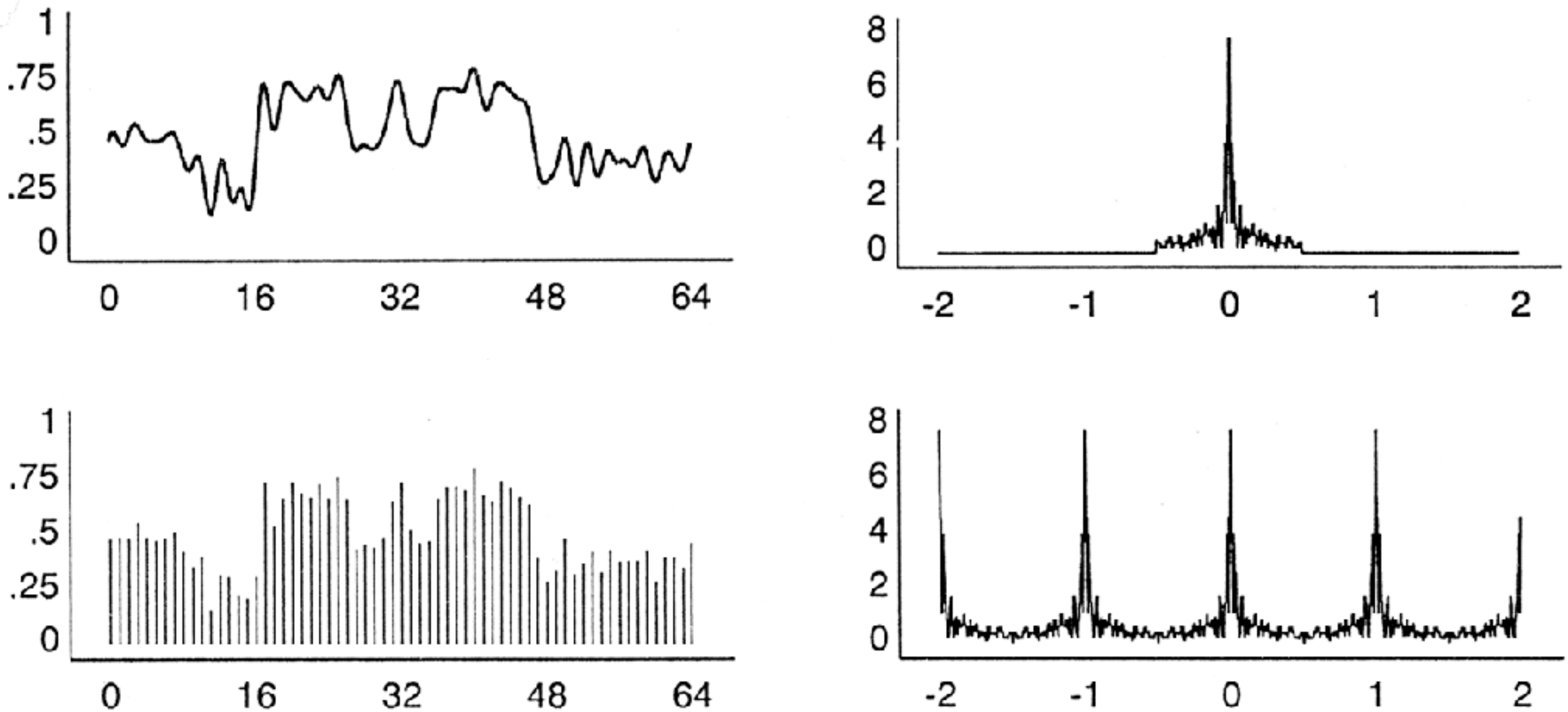
Reconstruction Examples

- Sampling and Reconstruction of a scanline

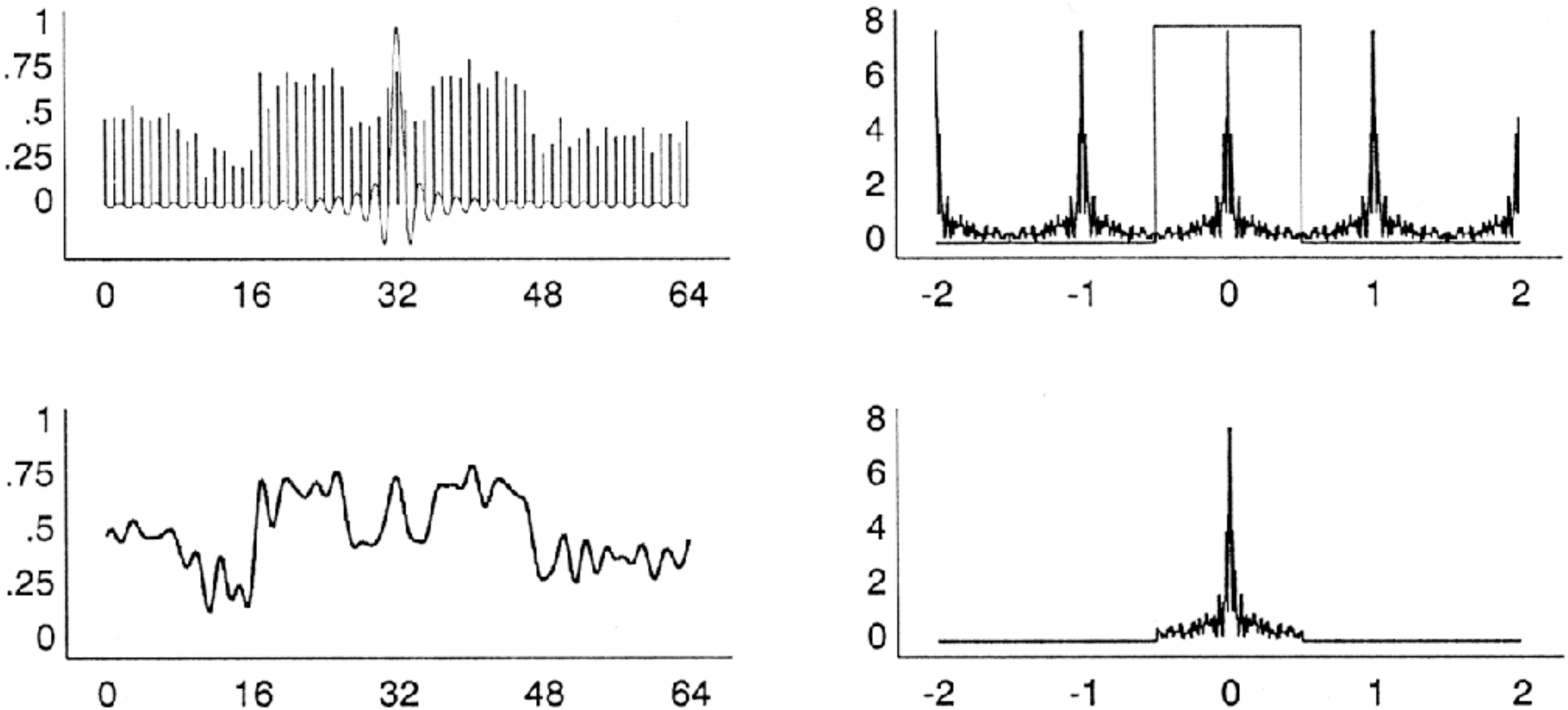
- With sufficient bandwidth
- With insufficient bandwidth
- With band limiting

- With sinc() and tent kernels

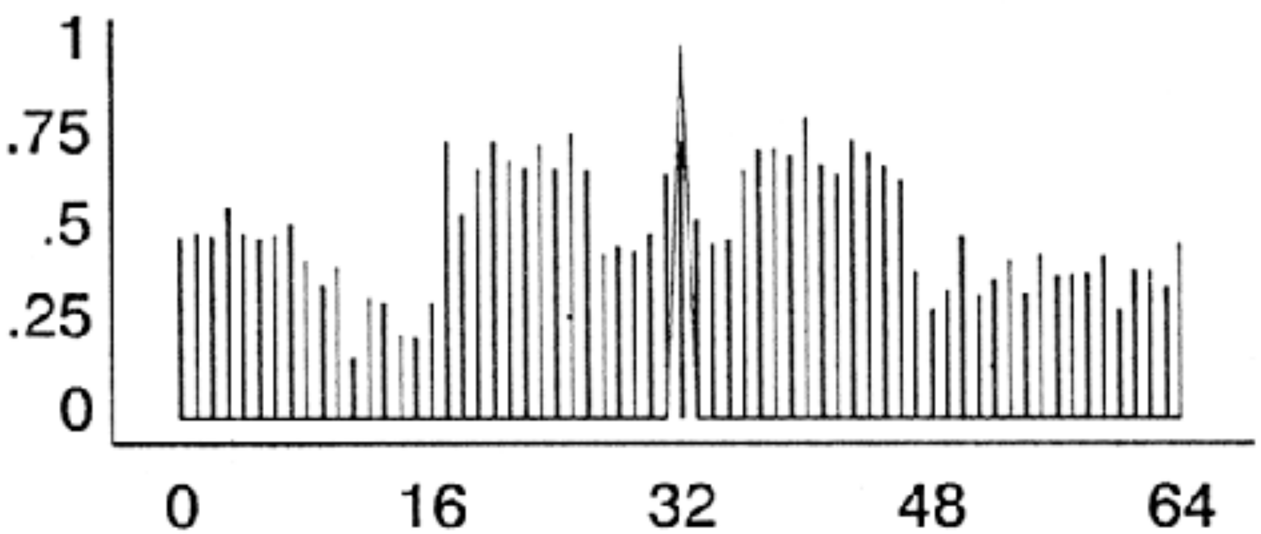
Sampling > w

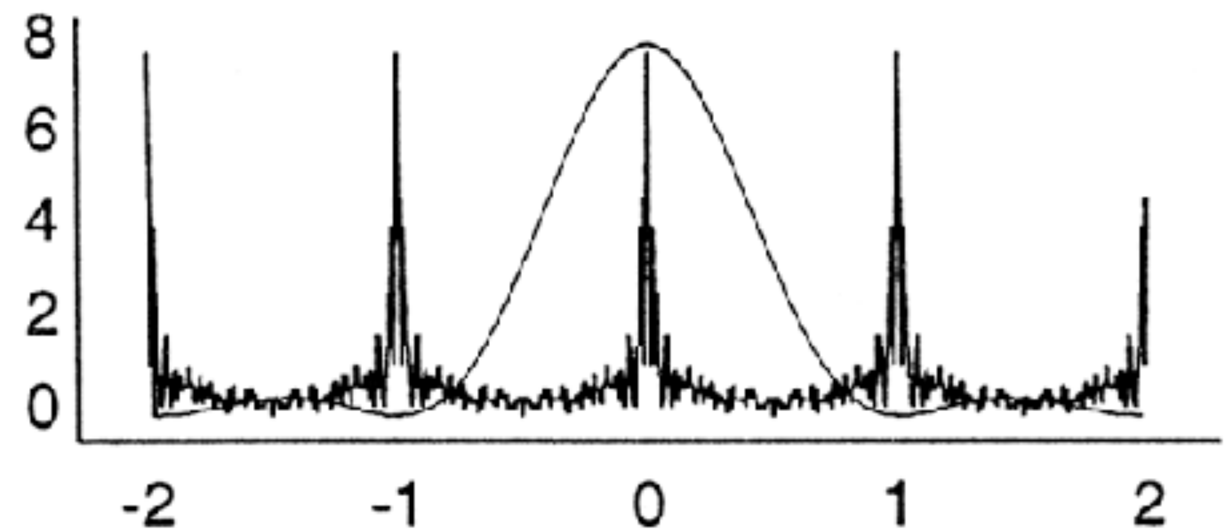


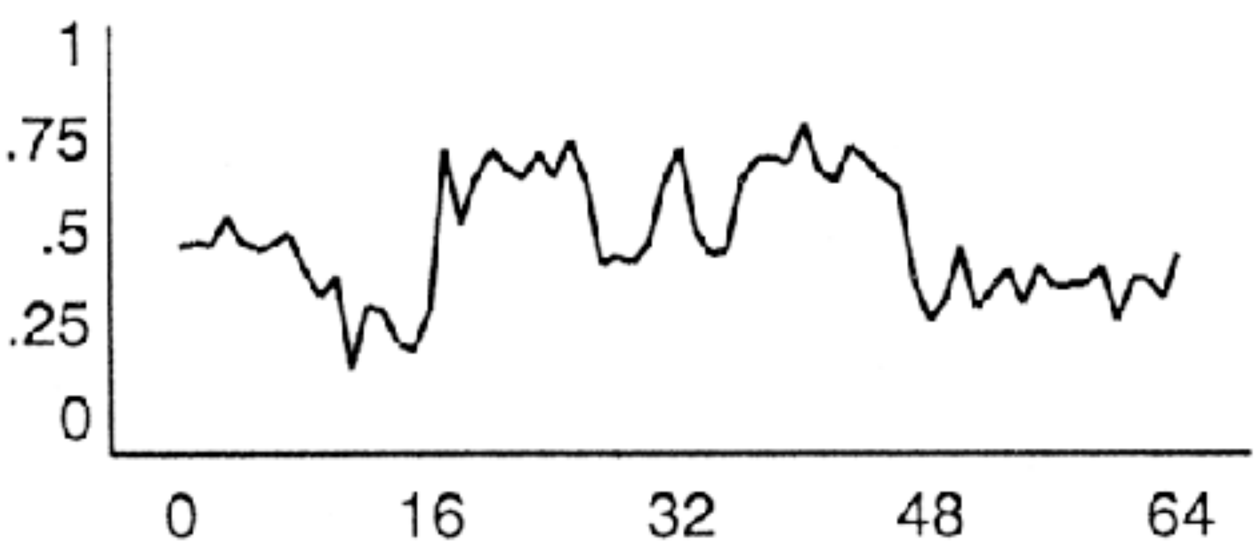
Sampling > w: Sinc() kernel

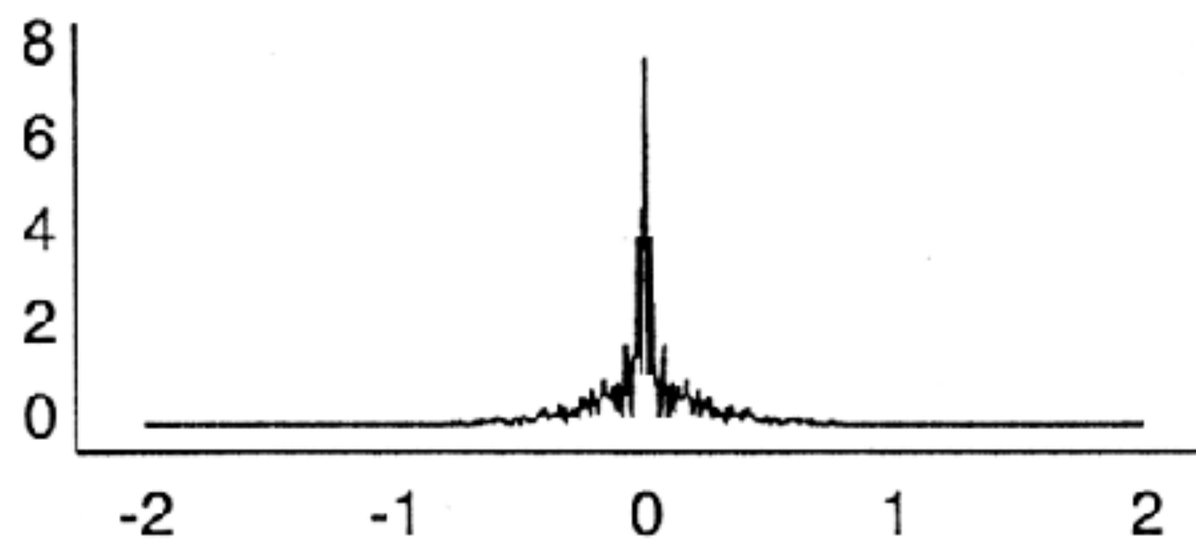


Sampling > w: Tent Kernel

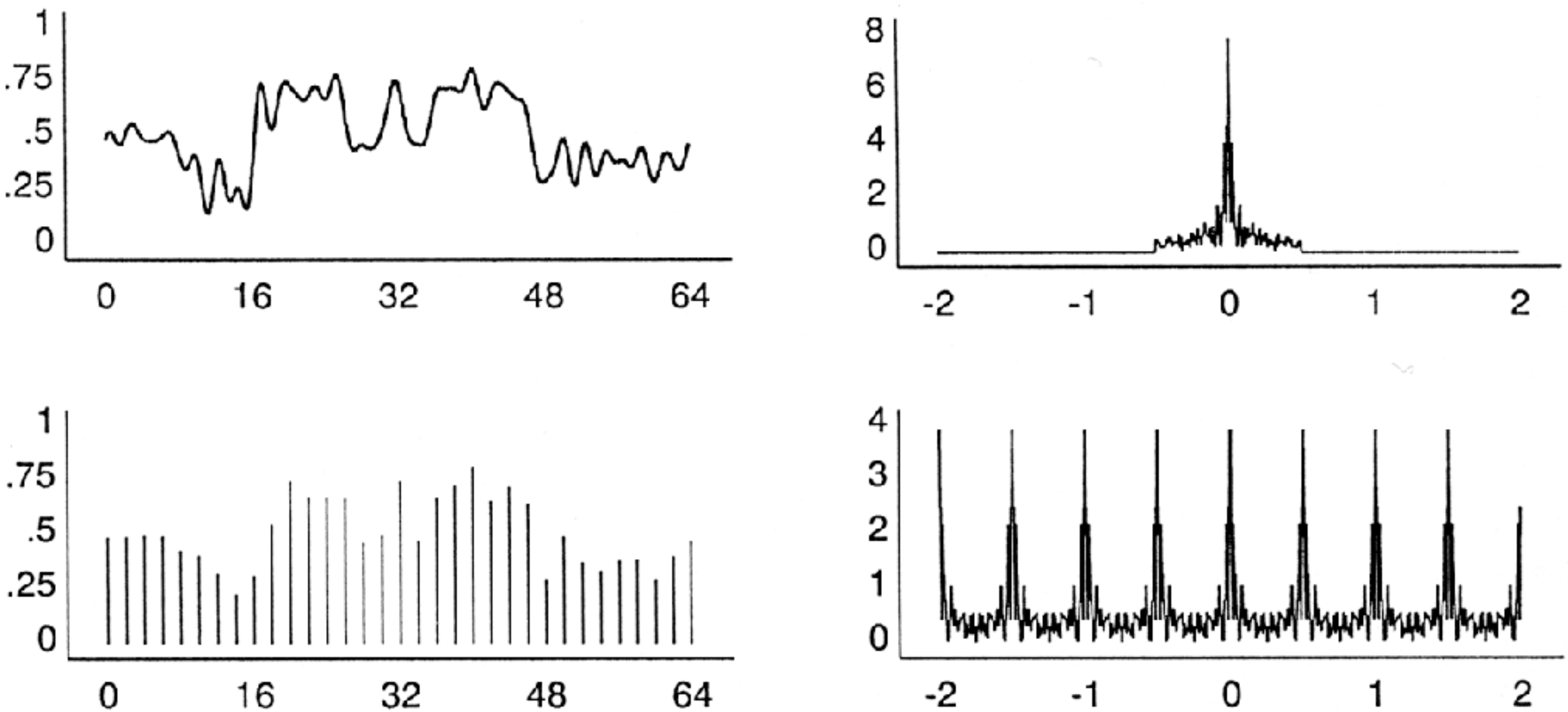








Sampling < w


1
.75
.5
.25
0
0
16
32
48
64
8
6
4
2
0
-2
-1
0
1
2
1
.75
.5
.25
0
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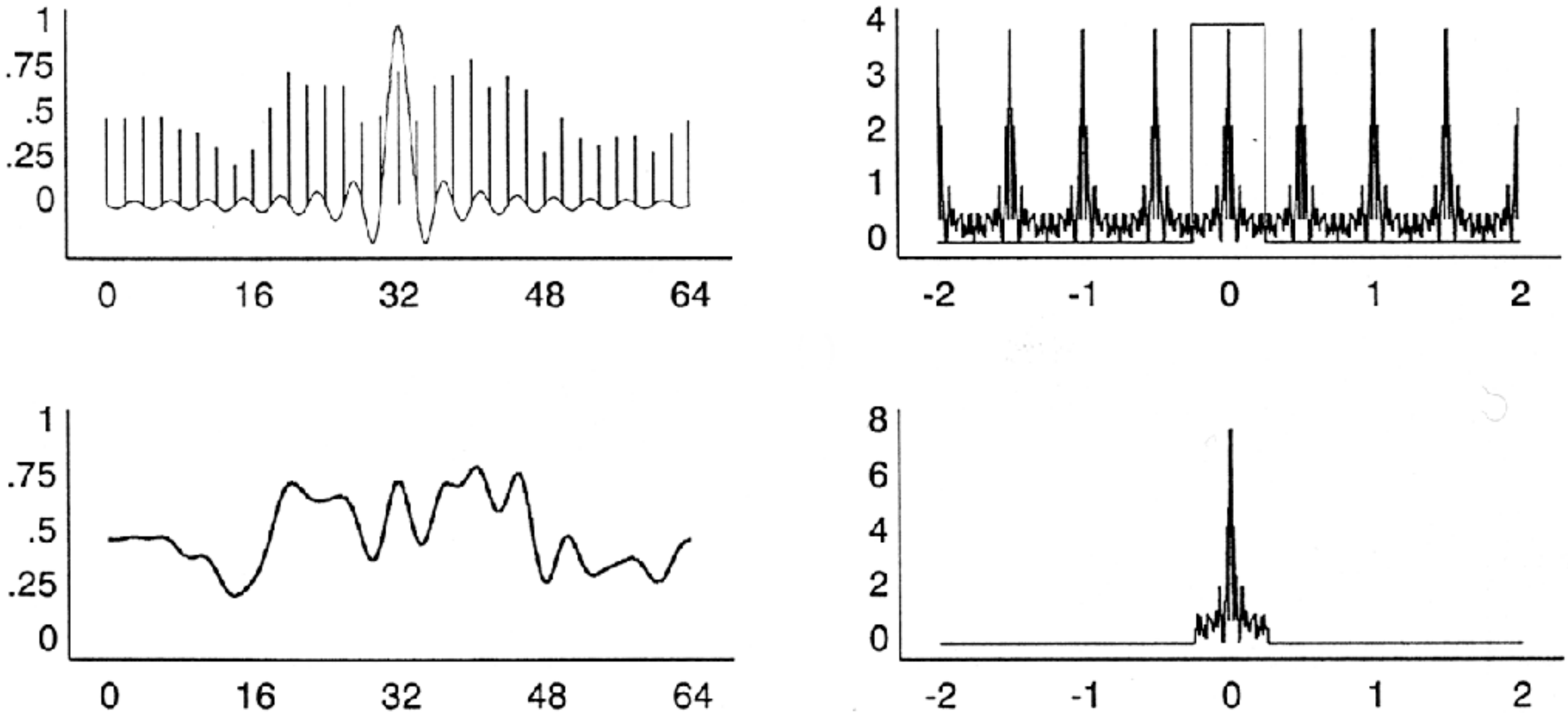
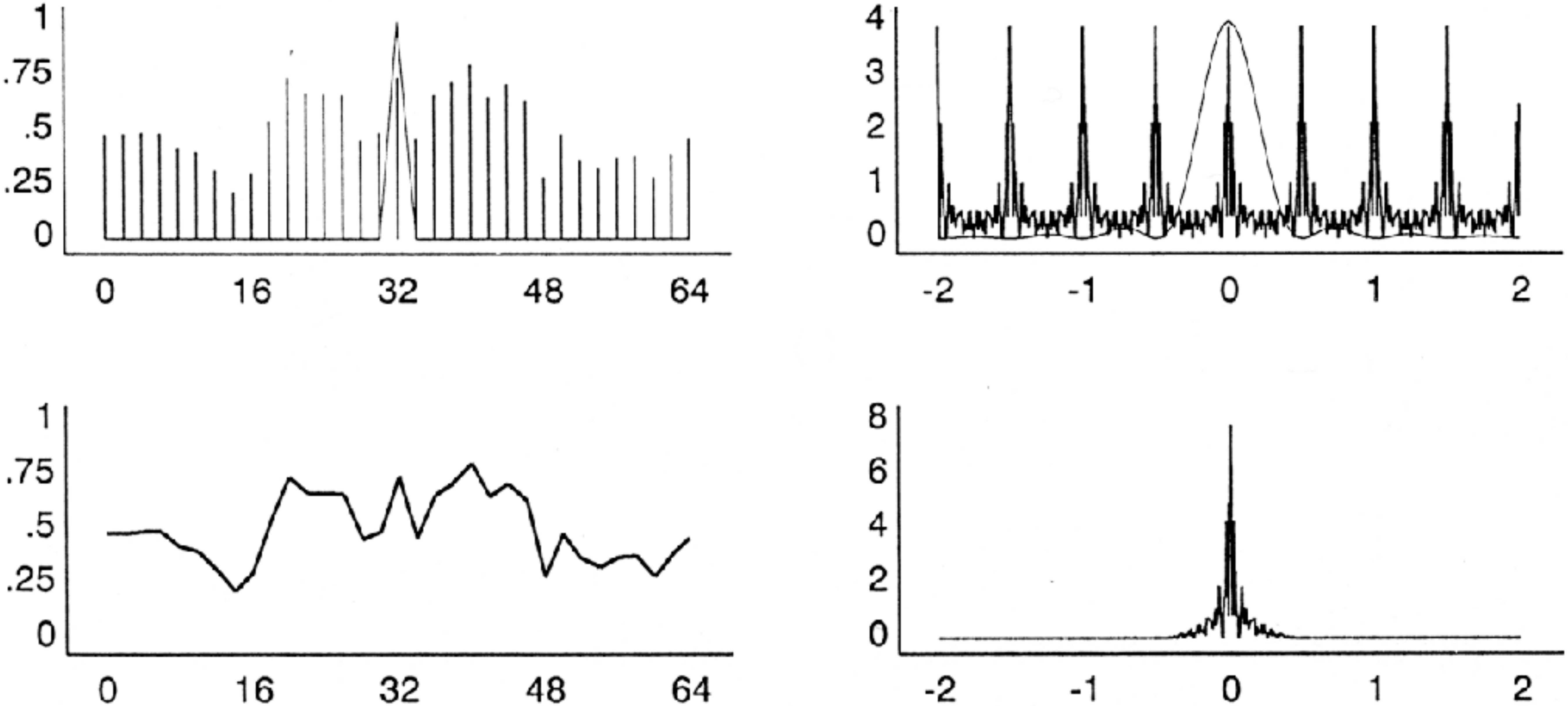
32
48
64
4
3
2
1
0
-2
-1
0
1
2

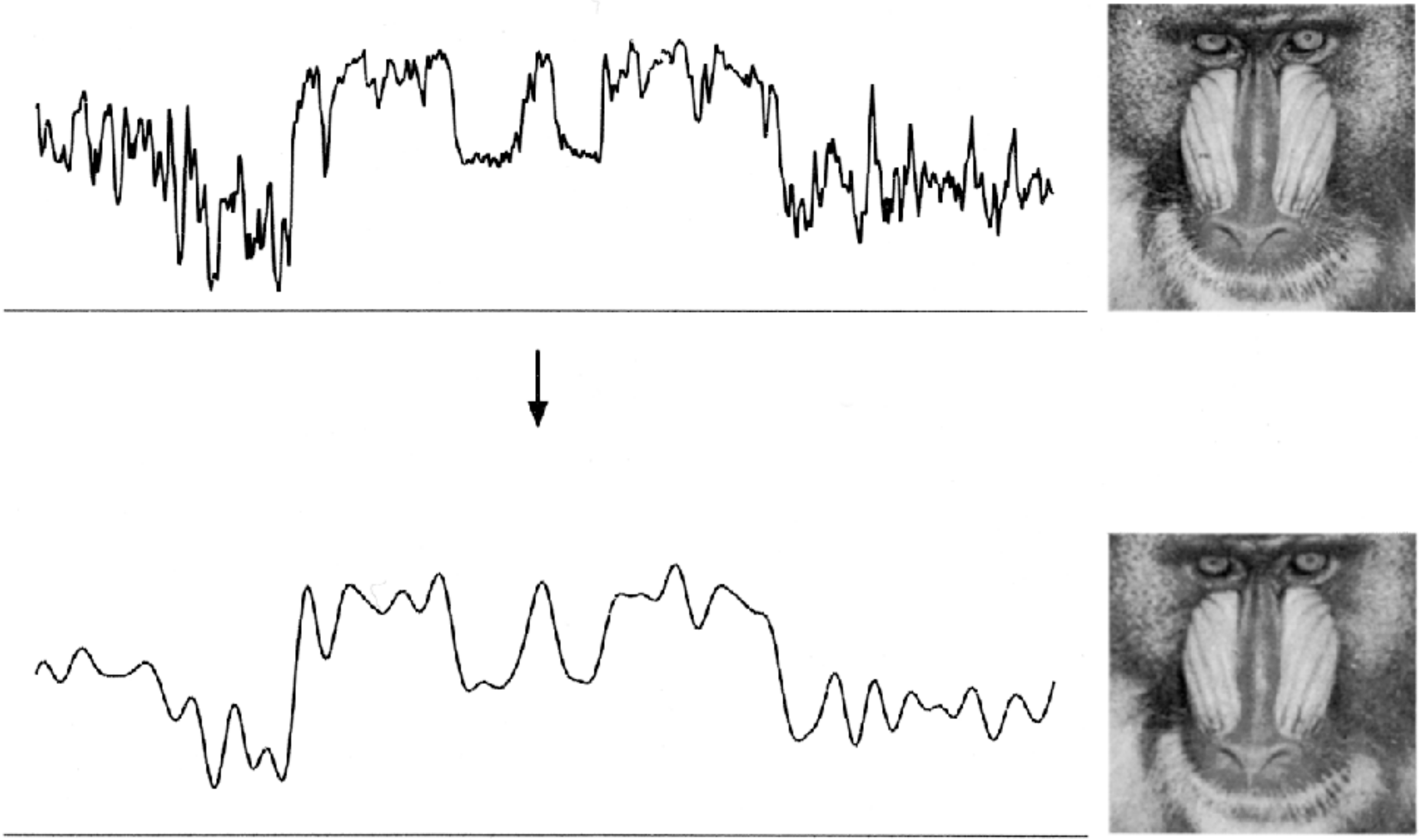
Sampling < w: Sinc()



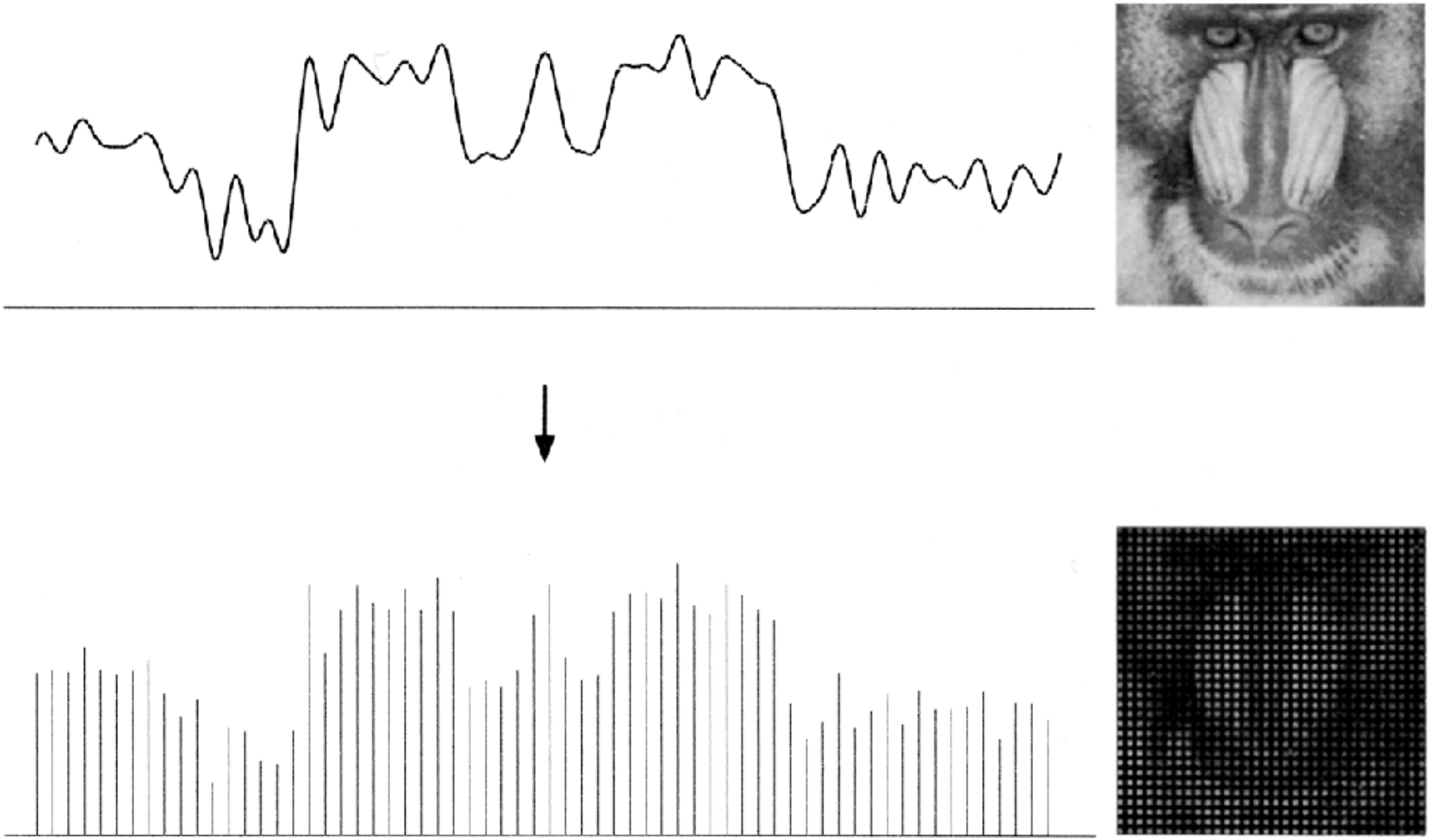
Sampling < w: Tent Kernel



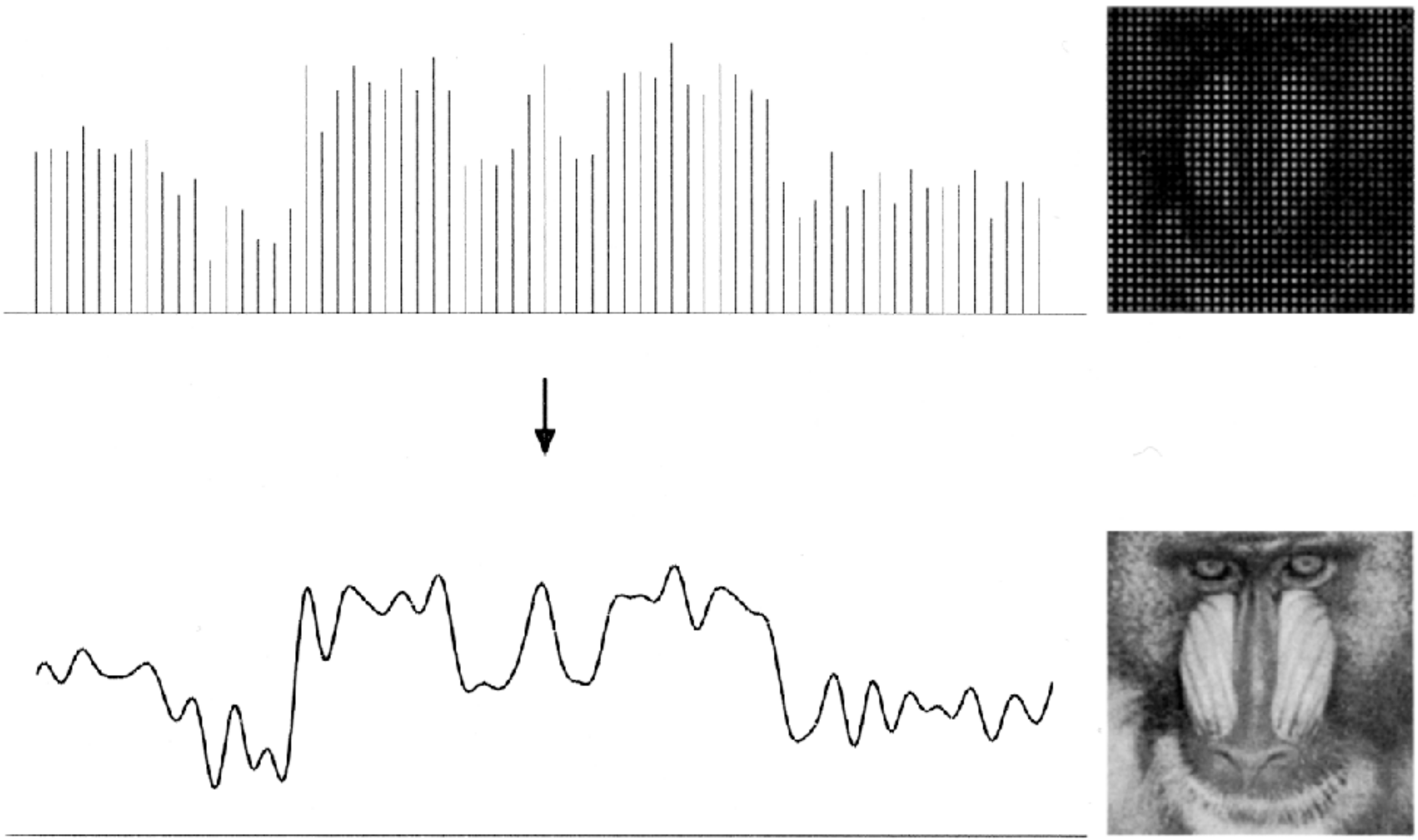
Band Limiting the Signal



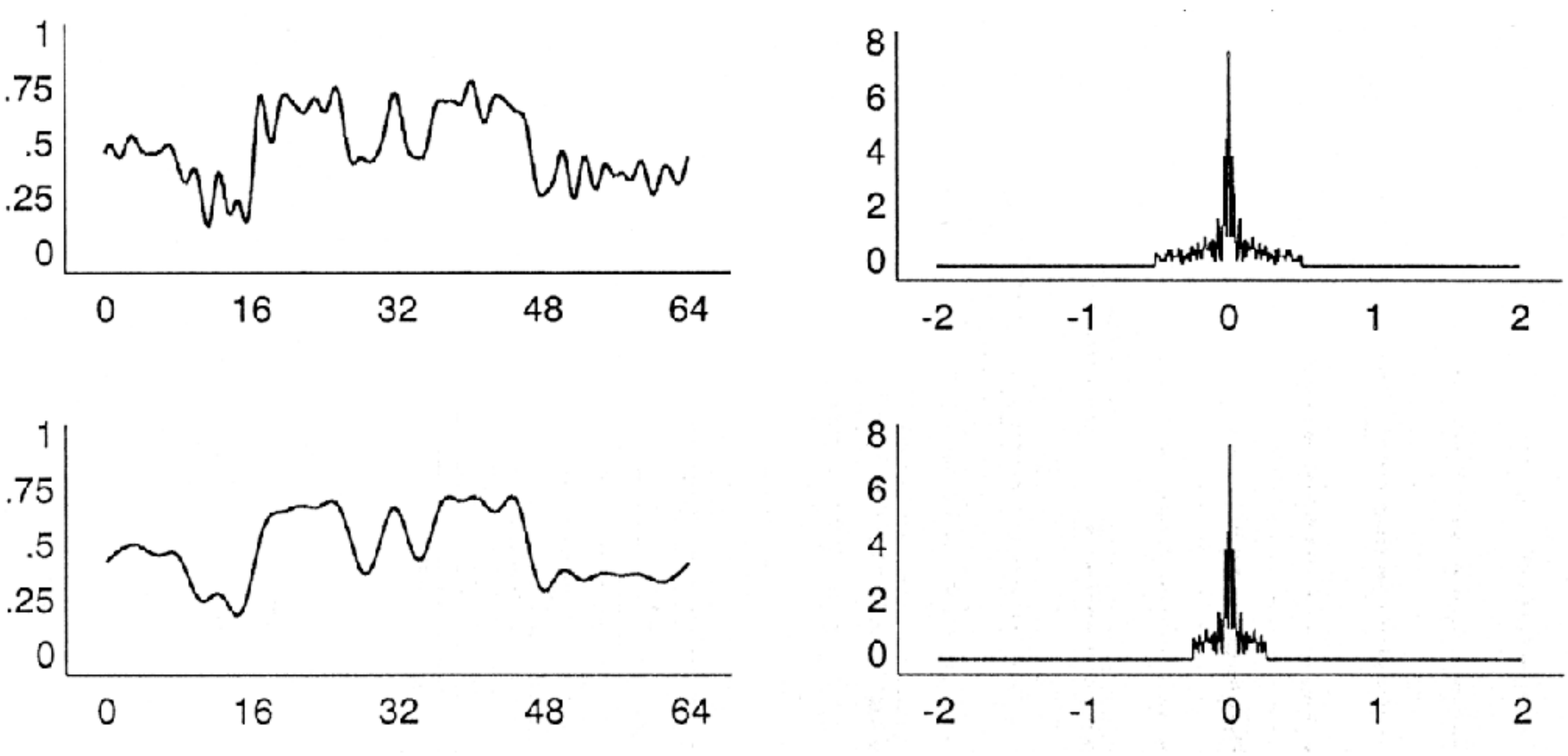
Band Limiting the Signal



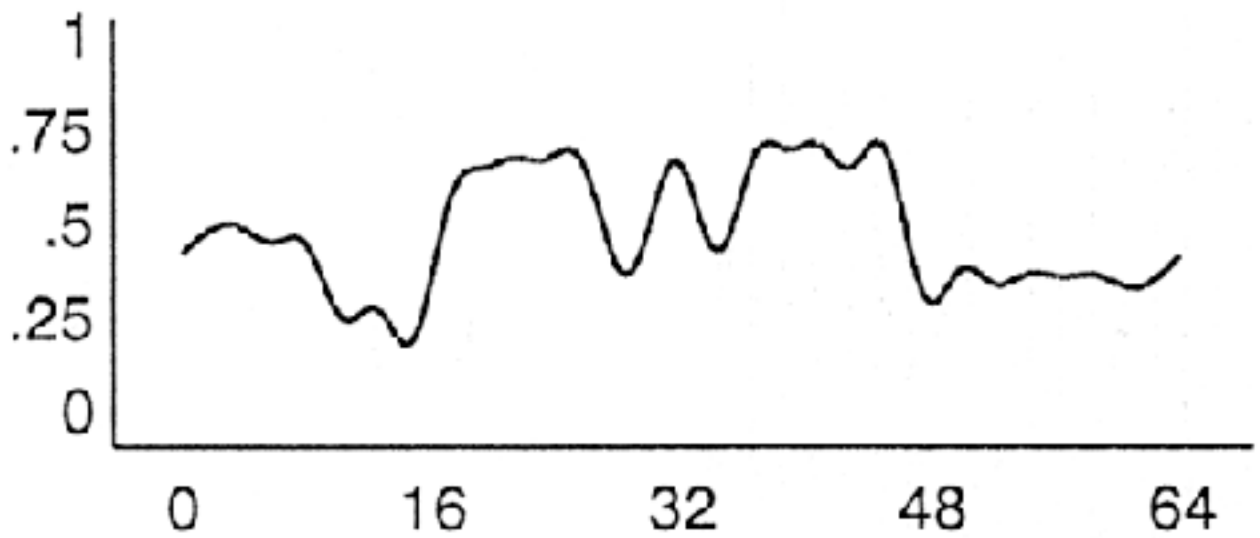
Band Limiting the Signal

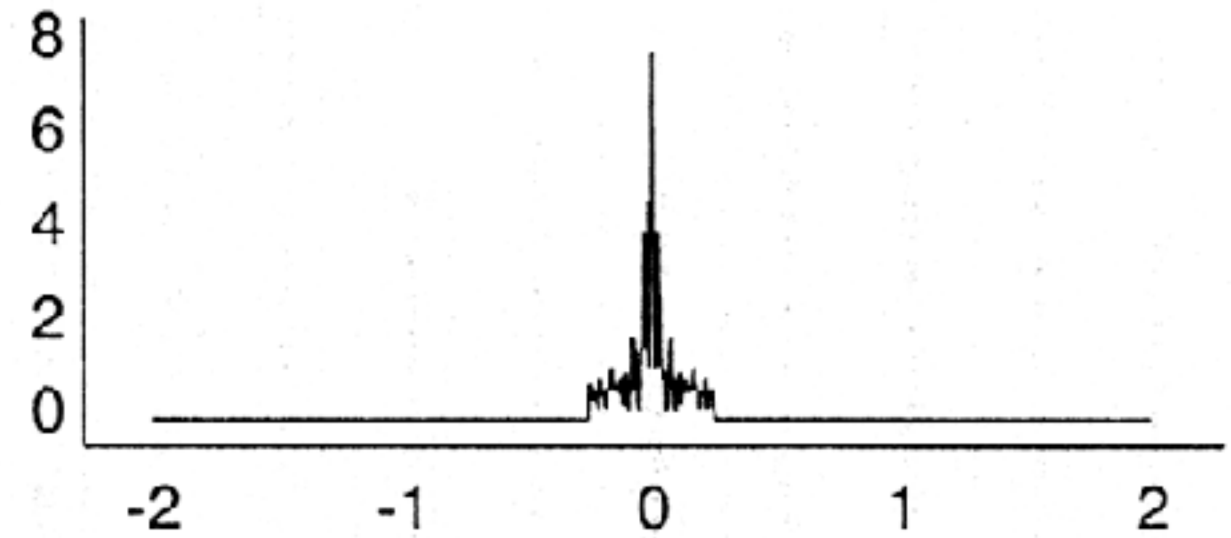


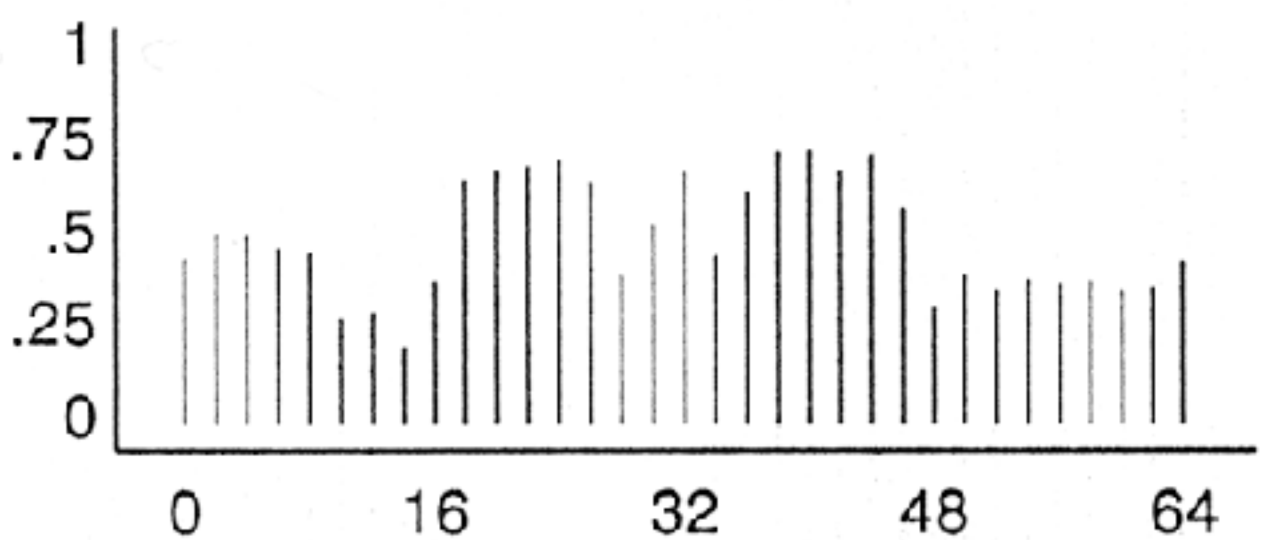
Band Limiting

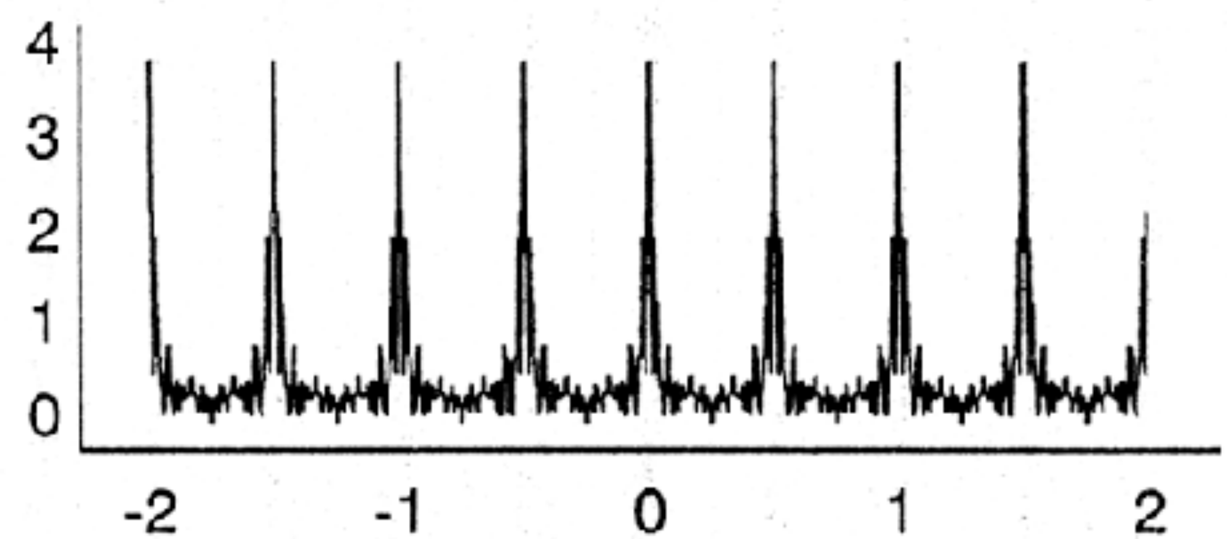


Sampling

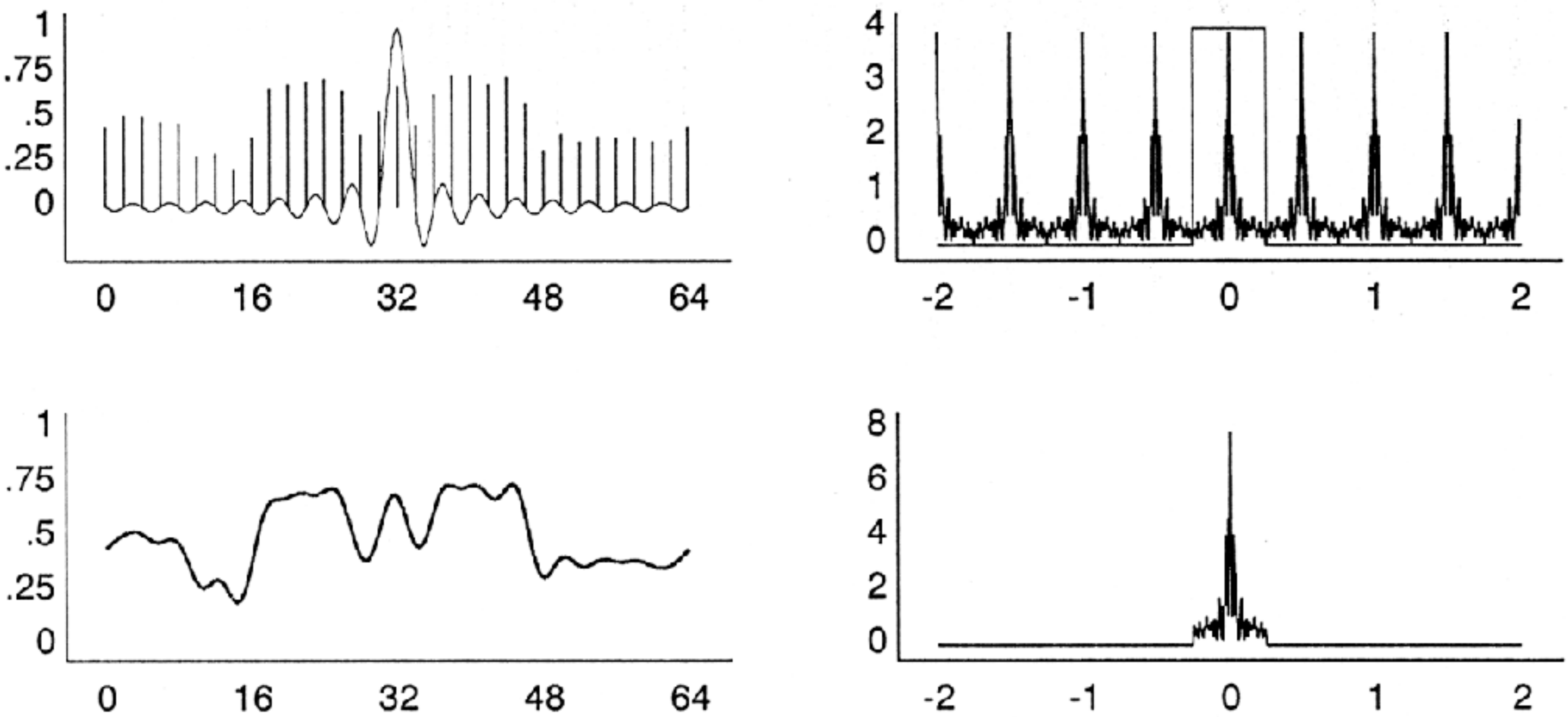

1
.75
.5
.25
0
0
16
32
48
64


8
6
4
2
0
-2
-1
0
1
2


1
.75
.5
.25
0
0
16
32
48
64


4
3
2
1
0
-2
-1
0
1
2

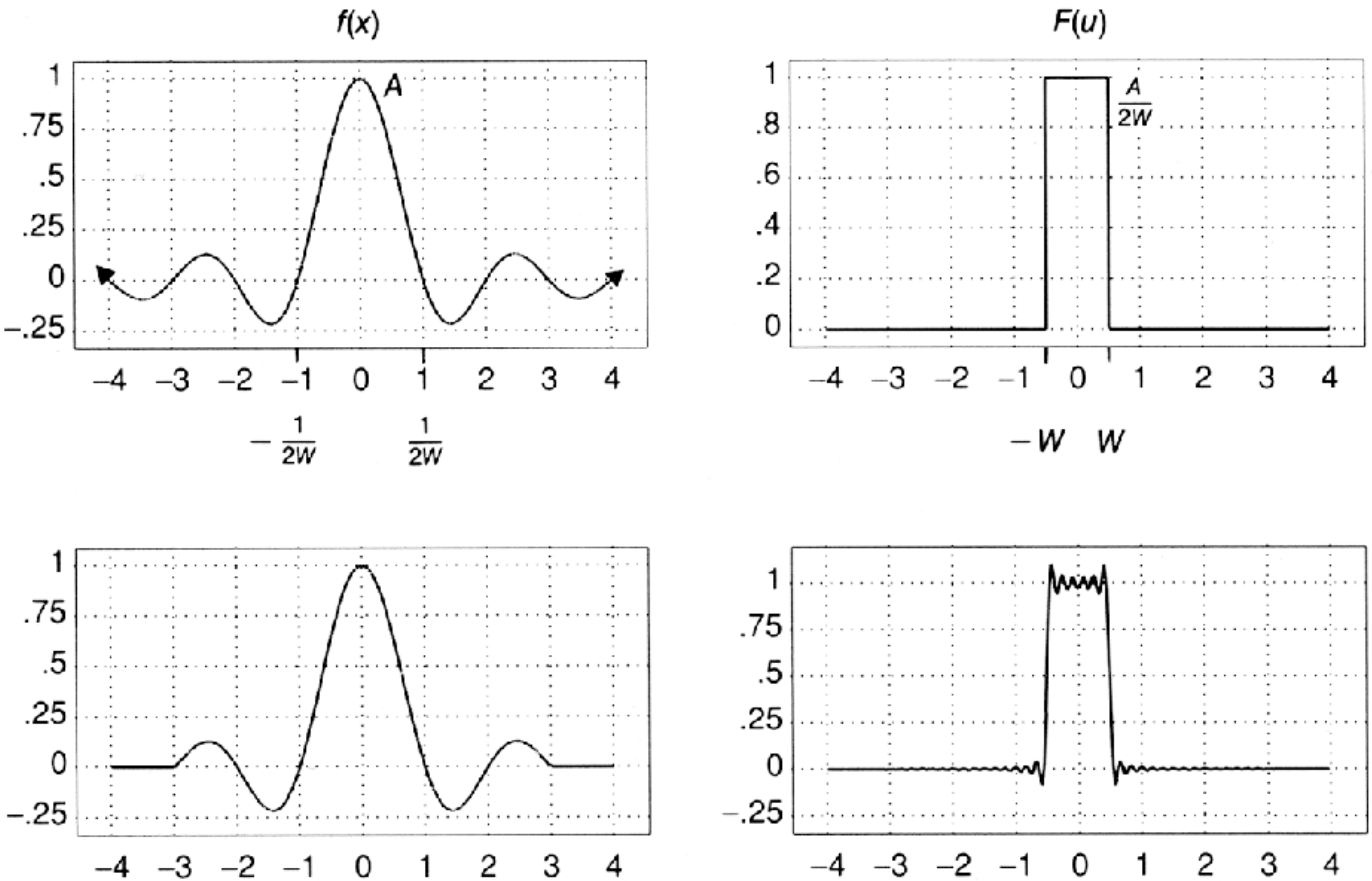
Reconstruction



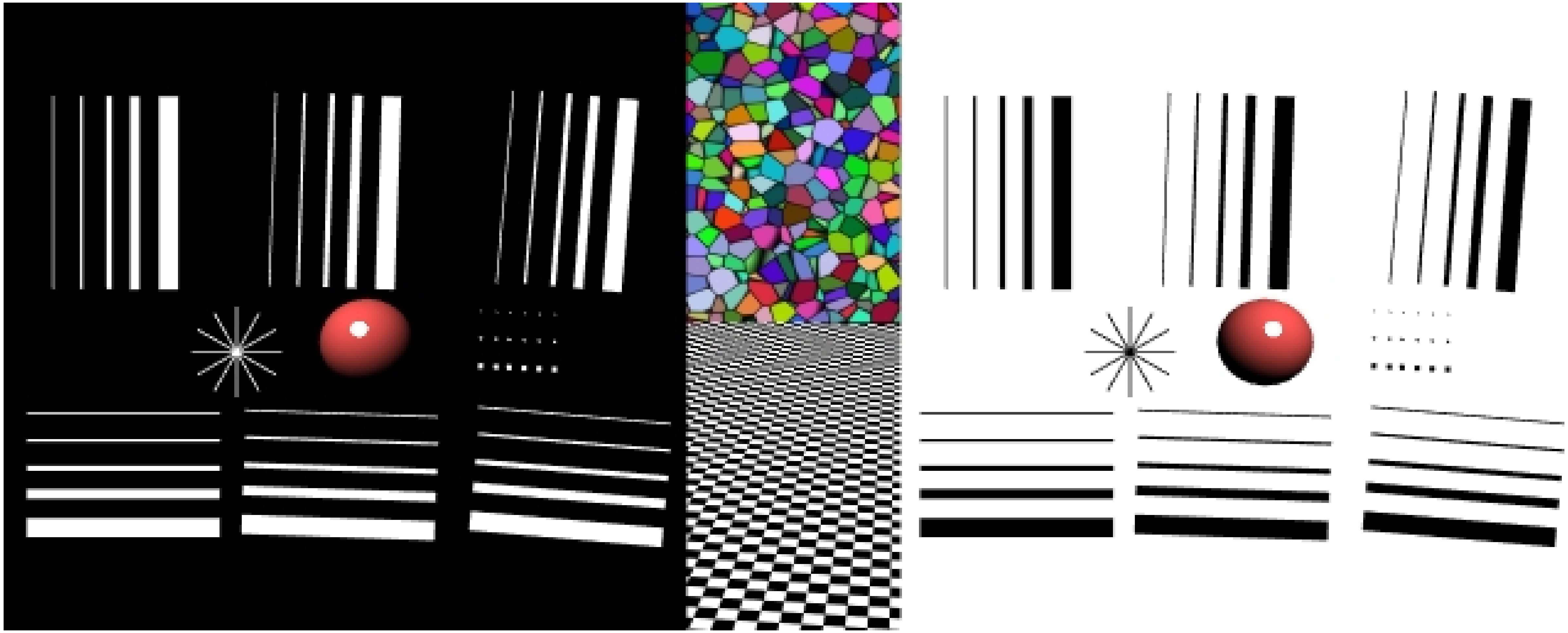
Reconstruction in Practice

- An ideal reconstruction filter exists:
  Sinc(x)
- Problem: Sinc(x) is unusable in practice, as it has infinite extent
- And to add insult to injury, truncated Sinc(x) is actually worse than many alternatives
- In practice, one uses a number of sub-optimal filters, depending on application

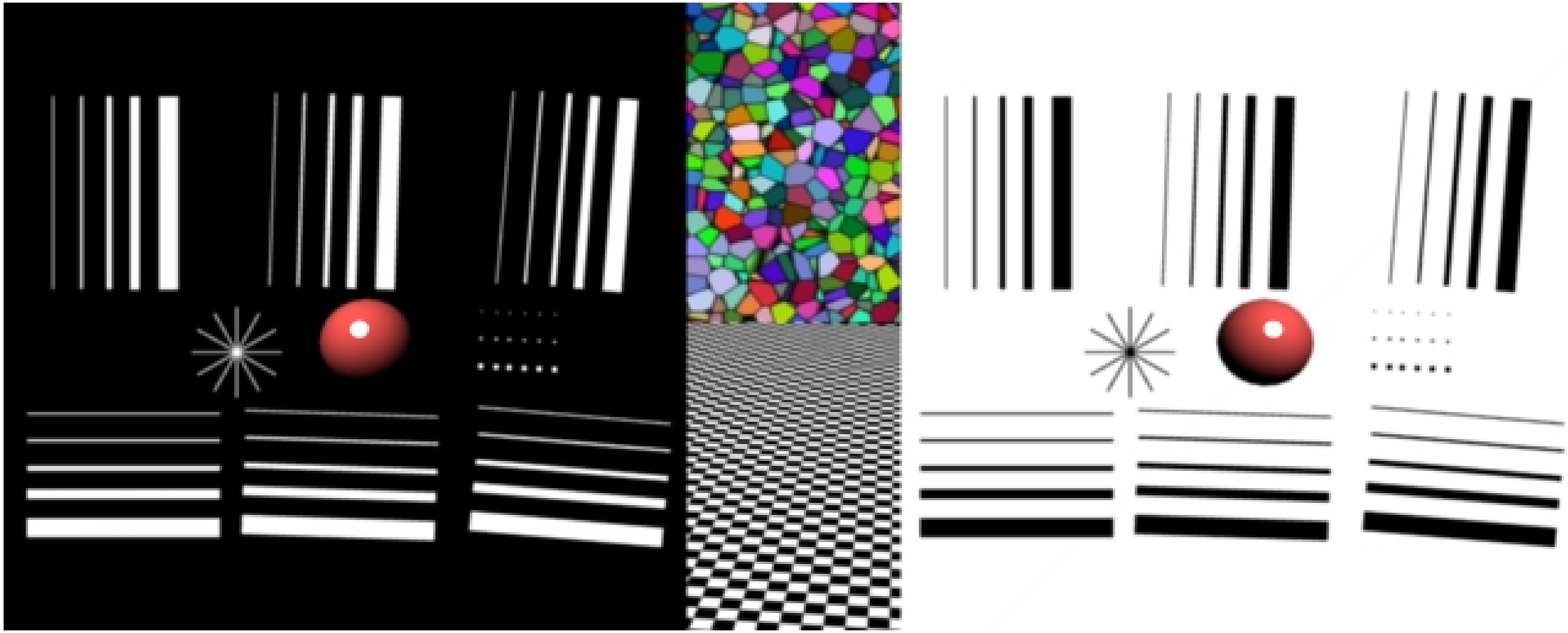
Sinc() & cut-off Sinc()



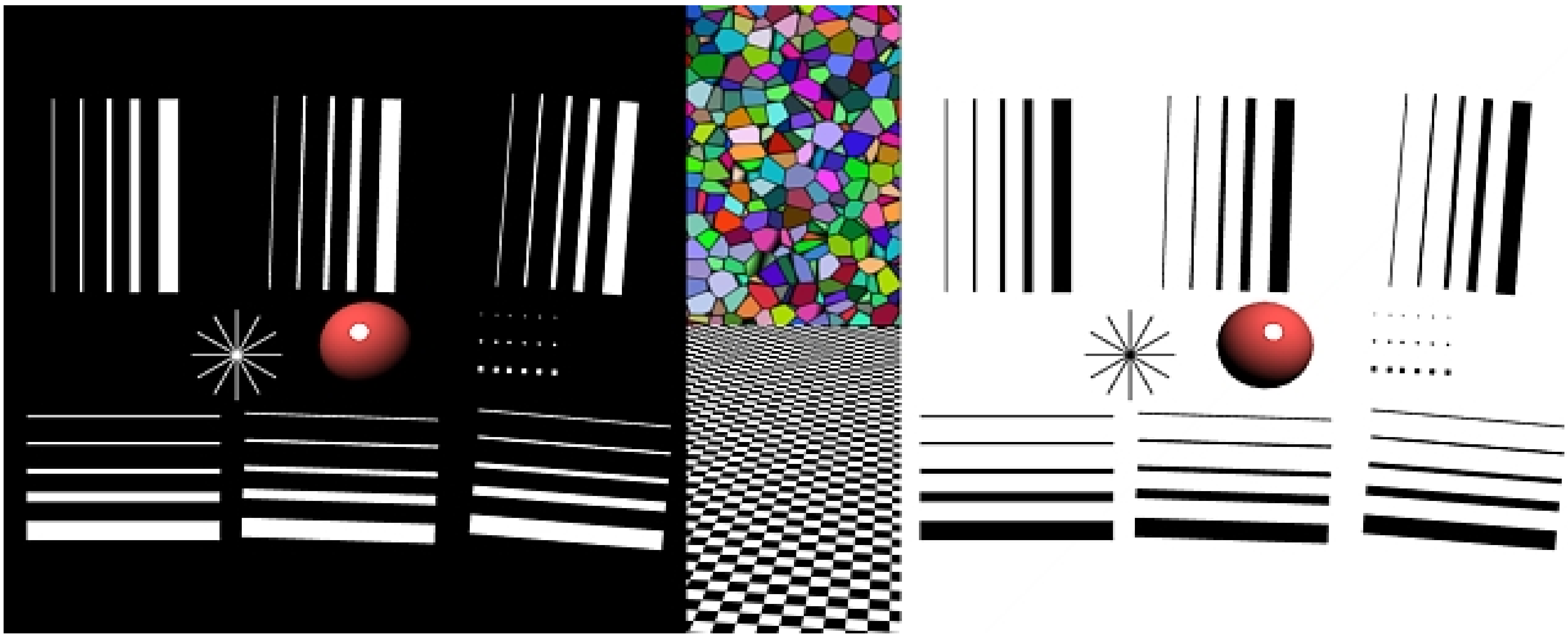
Blender



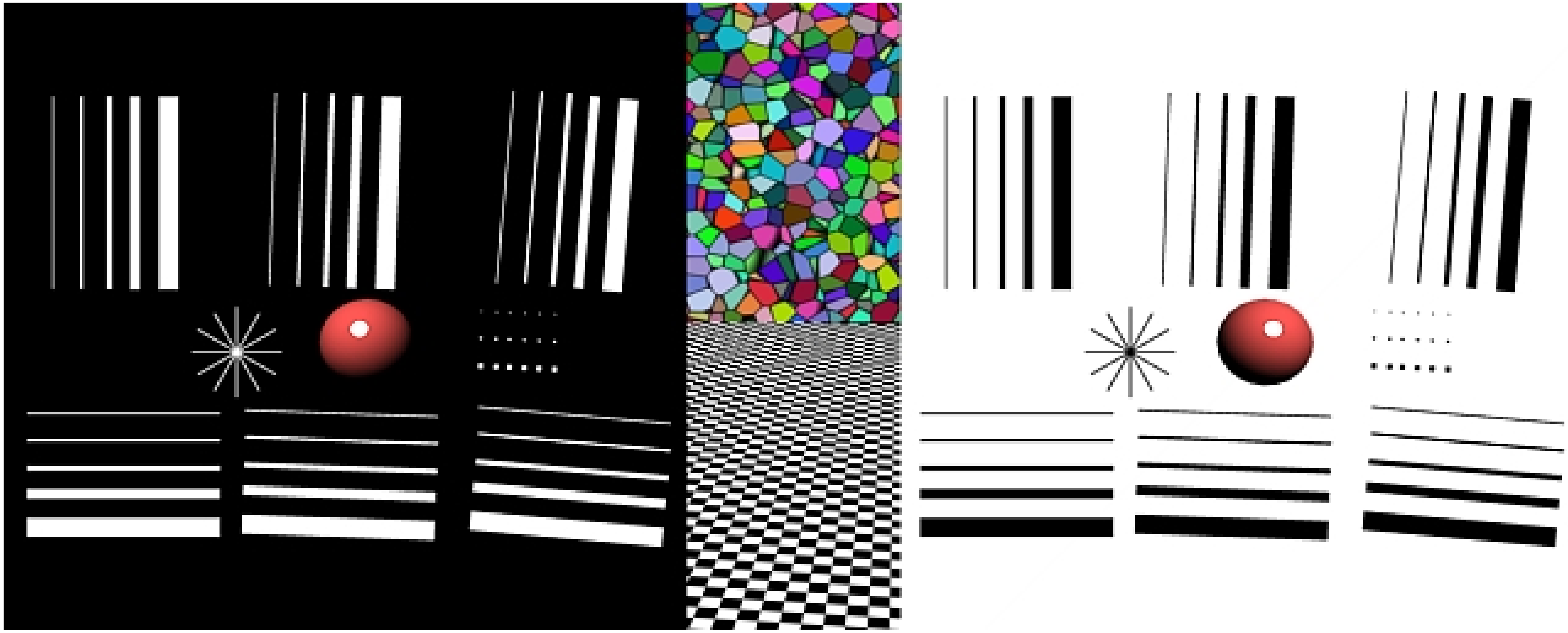
Blender



Blender



Blender



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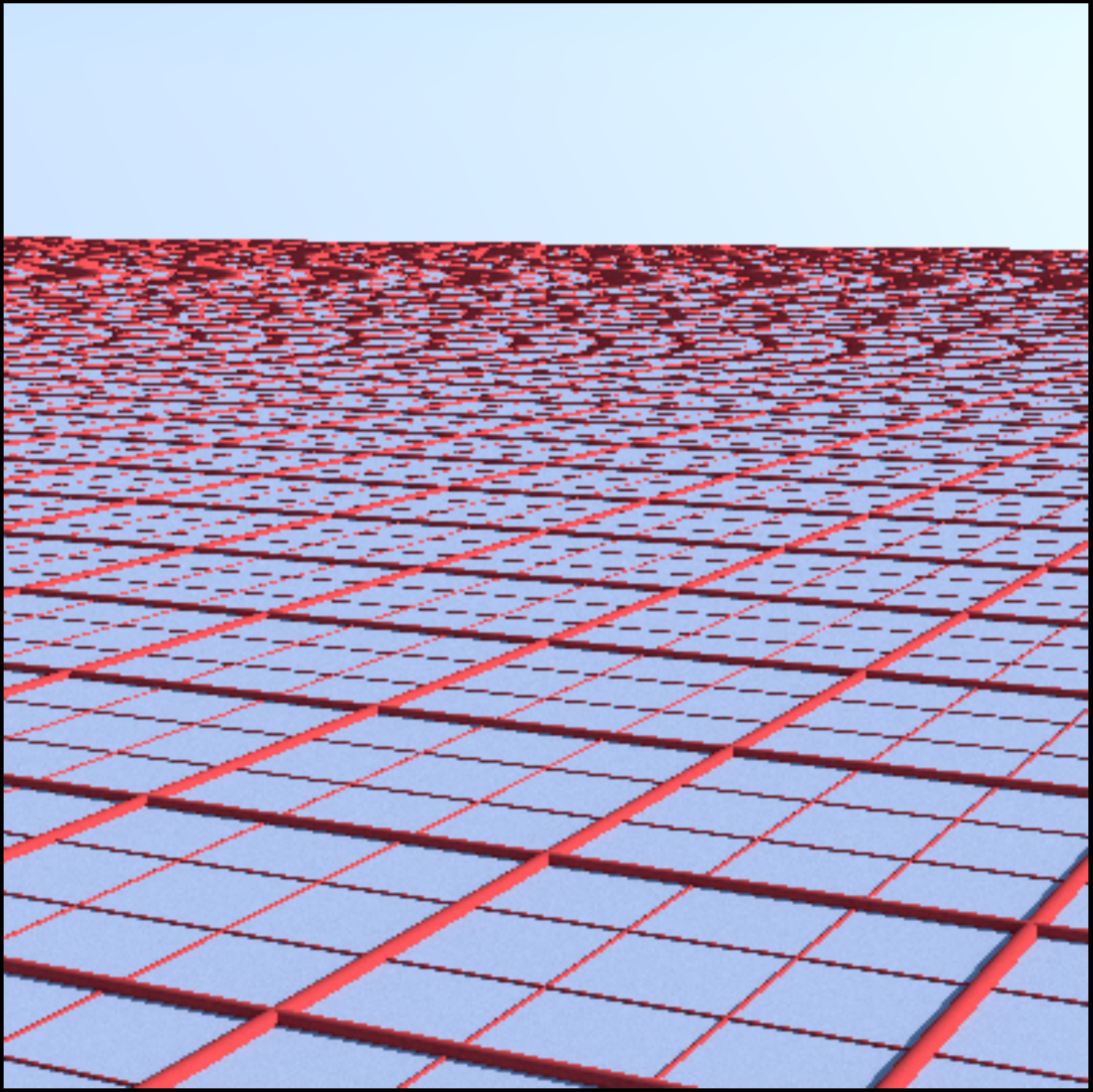
Grid Example



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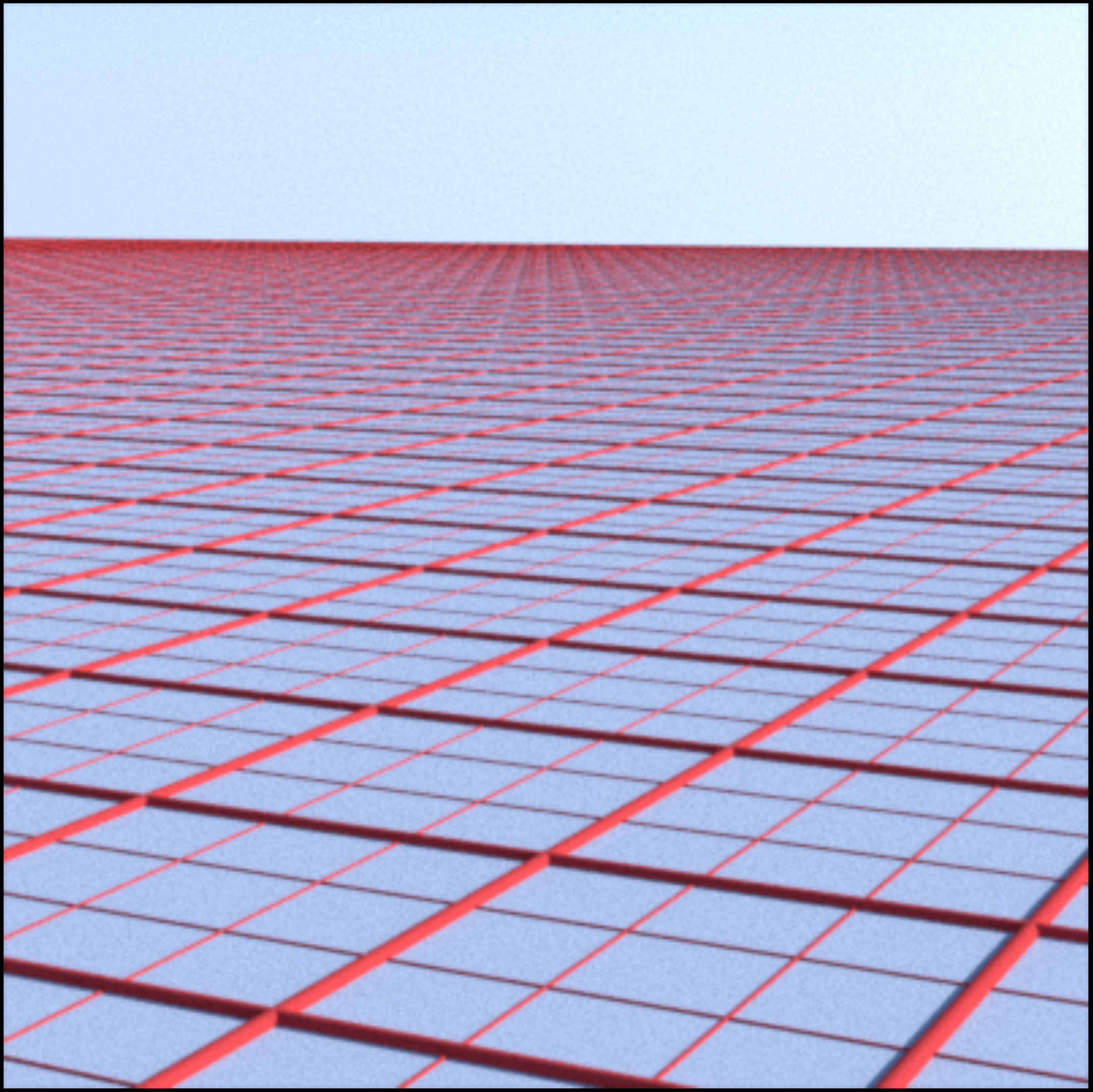
Grid Example



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Grid Example